Foundations of a Spacetime Path Formalism for Relativistic Quantum Mechanics

Ed Seidewitz[*]

14000 Gulliver's Trail, Bowie MD 20720 USA

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Abstract

Quantum field theory is the traditional solution to the problems inherent in melding quantum mechanics with special relativity. However, it has also long been known that an alternative first-quantized formulation can be given for relativistic quantum mechanics, based on the parametrized paths of particles in spacetime. Because time is treated similarly to the three space coordinates, rather than as an evolution parameter, such a spacetime approach has proved particularly useful in the study of quantum gravity and cosmology. This paper shows how a spacetime path formalism can be considered to arise naturally from the fundamental principles of the Born probability rule, superposition, and Poincaré invariance. The resulting formalism can be seen as a foundation for a number of previous parametrized approaches in the literature, relating, in particular, "off-shell" theories to traditional on-shell quantum field theory. It reproduces the results of perturbative quantum field theory for free and interacting particles, but provides intriguing possibilities for a natural program for regularization and renormalization. Further, an important consequence of the formalism is that a clear probabilistic interpretation can be maintained throughout, with a natural reduction to non-relativistic quantum mechanics.

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[*]Electronic address: seidewitz@mailaps.org

I. INTRODUCTION

The idea of constructing quantum states as a "sum over histories" is well known in the form of the Feynman path integral formulation. However, this approach is best known in its application to non-relativistic quantum mechanics [1, 2], in which particle paths are parametrized by coordinate time. A natural relativistic generalization is to consider parametrized paths in four-dimensional spacetime rather than time-parametrized paths in three-dimensional space. Feynman himself developed such an approach, and this conception seems to have informed much of Feynman's early view of relativistic quantum mechanics [3, 4, 5].

At an even earlier date, Stueckelberg presented a detailed formulation of relativistic quantum mechanics in terms of parametrized spacetime paths [6, 7]. A number of other authors (notably [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]) have also developed related approaches involving an invariant "fifth parameter" governing the evolution of a quantum system, though not necessarily identifying this explicitly as a path parameter.

A key feature of these approaches is that time is treated comparably to the three space coordinates, rather than as an evolution parameter. This is particularly applicable to the study of quantum gravity and cosmology, in which the fundamental equations (such as the Wheeler-DeWitt equation) make no explicit distinction for the time coordinate (see, e.g., [19, 20, 21, 22, 23, 24, 25]).

Also, in the infinite-tension limit, string theory reduces to a worldline formalism for relativistic quantum theory [26, 27, 28, 29, 30, 31, 32]. One would therefore expect a path formulation of relativistic quantum mechanics to provide a natural bridge to the typically first-quantized formulation of string theory.

Despite the promise of the approach, spacetime path formalisms have often been presented in the literature as simply alternative formulations of results obtained from the more traditional quantum field theory formalism. The motivation of the present paper, however, is to construct a first-quantized spacetime path formalism that can be considered foundational in its own right. This means that many typical tools of field theory, such as Hamiltonian dynamics and the Lagrangian stationary action principle for fields, cannot be assumed to apply *a priori*.

Instead, we will begin with the fundamental principles of special-relativistic quantum

theory—the Born probability rule, superposition, and Poincaré invariance—and introduce six additional, physically motivated postulates related to spacetime paths. (The perhaps even more fundamental question of why quantum probabilities are given via superpositions of probability amplitudes will not be addressed here.) Results deduced from these postulates then provide the basis for further physical interpretation.

Since this formalism is first quantized, particular care is given to properly handling particles and antiparticles and to developing a consistent probabilistic interpretation. The result is an approach that fully deals with the usual issues of negative energies and negative probabilities, but without necessitating the introduction of fields as fundamental entities. Rather, fields can be considered to be simply a convenient formalism for handling multiparticle states. The present work only discuses massive scalar particles, but the approach can be extended to handle non-scalar particles (e.g., [33, 34]).

Section II first introduces the formalism for free scalar particles, culminating in free multiparticle fields. Section III then extends the formalism to consider interacting states and scattering. In order to reduce clutter in the text, certain propositions resulting from purely mathematical, but somewhat involved, derivation are cited without proof in the main body of the text, with proofs given in appendices.

Natural units with $\hbar = 1 = c$ are used throughout the following and the metric has a signature of $(-+++)$.

II.   FREE PARTICLES

For any path based approach, it is obviously critical to be clear on what is meant by the term *path*. In the present case, a *path* for a particle is an arbitrary curve through spacetime, that is, a continuous (though not necessarily differentiable), one-dimensional subspace of spacetime. Note that there is no *a priori* requirement that such a curve is timelike or lightlike. Indeed, the path may cross arbitrarily forwards and backwards in time. Since such a path is continuous, there is a one-to-one mapping between it and some interval of the real numbers. That is, a path may be given by functions $q^\mu(\lambda)$, for $\mu = 0, 1, 2, 3$, of a *path parameter* λ.

In this formulation, the path parameter λ serves a purpose similar to that of time in the traditional non-relativistic path integral approach [2]. For the restricted case of an

everywhere-timelike path, this parameter is analogous to proper time. For the general case of an unrestricted path, there has been some debate as to the physical nature of the path parameter (see, for example, [35, 36]). In order not to presuppose any specific interpretation, we will consider, for each path, all possible parameterizations of the path.

To do this, choose a fiducial parametrization s, say over the interval $[0, 1]$, and define any other parametrization as a monotonically increasing function $\lambda(s)$. Geometrically, the so called *lapse multiplier*

$$w(s) \equiv \frac{\mathrm{d}\lambda}{\mathrm{d}s} > 0$$

then gives an effective length metric $\mathrm{d}\lambda = w(s)\mathrm{d}s$ for the path, and the corresponding parametrization λ is an *intrinsic length* measure along the path.

Given this basic conception of a particle path, this section will review the fundamental postulates required for a path integral approach, derive the scalar free particle propagator and carefully consider the corresponding probability interpretation.

A. The Free Particle Propagator

The fundamental postulate of any spacetime path integral approach is that a particle's transition amplitude between two points in spacetime is a superposition of the transition amplitudes for all possible paths between those points. Let the functional $\Delta[q]$ give the transition amplitude for a path $q^\mu(\lambda)$. Then the total transition amplitude $\Delta(x, x_0)$ must be given by a path integral over $\Delta[q]$, for all paths q from x_0 to x.

Postulate 1. *For a free scalar particle, the transition amplitude $\Delta(x, x_0)$ is given by the superposition of path transition amplitudes $\Delta[q]$, for all possible 4-dimensional path functions $q^\mu(\lambda)$ beginning at x_0 and ending at x, parametrized by all possible monotonically increasing functions $\lambda(s)$. That is,*

$$\Delta(x, x_0) = \int \mathrm{D}\lambda \, \theta\left[\frac{\mathrm{d}\lambda}{\mathrm{d}s}\right] F[\lambda]\Delta(x, x_0; [\lambda]) \,, \tag{1}$$

where the real-valued functional $F[\lambda]$ allows for the possibility of different weights for different parametrizations, and

$$\Delta(x, x_0; [\lambda]) \equiv \eta[\lambda] \int \mathrm{D}^4 q \, \delta^4(q(\lambda(1)) - x)\delta^4(q(\lambda(0)) - x_0)\Delta[q] \,, \tag{2}$$

where $\eta[\lambda]$ is a parametrization-dependent normalization factor as required to keep the path integral finite.

Note that, in Eqs. (1) and (2), the notation $D\lambda$ indicates a path integral over the parameterization function $\lambda(s)$ while $D^4 q$ indicates a path integral over the four path functions $q^\mu(\lambda)$.

In the traditional Feynman sum-over-paths approach, the form of $\Delta[q]$ is simply assumed to be an exponential of the classical action [2]. This is justified because the resulting transition amplitudes agree with the results of the usual formulation of quantum mechanics. However, if the path-based formulation is to be considered foundational, one would prefer a more fundamental justification.

As a transition amplitude, $\Delta[q]$ strictly only applies to a particle on a specific path q from the starting position $q(\lambda_0)$ to the ending position $q(\lambda_1)$ (where the parameter range of q is $[\lambda_0, \lambda_1]$). However, by translational invariance in Minkowski spacetime, the particle propagation embodied in $\Delta[q]$ cannot depend on the absolute positions $q(\lambda)$, but only on the relative positions

$$\Delta q(\lambda) \equiv q(\lambda) - q(\lambda_0).$$

That is, we can take $\Delta[q] = \Delta[\Delta q]$.

Now, consider a *family* of parallel paths q_{x_0}, indexed by the starting position x_0, such that

$$q_{x_0}(\lambda) = x_0 + \Delta q(\lambda),$$

for a fixed relative position function Δq. Since all members of such a family have the same relative position function Δq, the amplitude $\Delta[q_{x_0}] = \Delta[\Delta q]$ must be the same for all members of the family.

Suppose that a probability amplitude $\psi(x_0)$ is given for a particle to be at an initial position x_0 and that the transition amplitude is known to be $\Delta[\Delta q]$ for a specific relative position function Δq. Then, the probability amplitude for the particle to traverse a specific path q_{x_0} from the family for relative position Δq is just $\Delta[q_{x_0}]\psi(x_0) = \Delta[\Delta q]\psi(x_0)$.

However, the very meaning of being on the specific path q_{x_0} is that the particle must propagate from the starting position at x_0 to the ending position at $q_{x_0}(\lambda_1)$. Therefore, the probability for reaching the end position $q_{x_0}(\lambda_1)$ must be the same as the probability for

having started out at the position x_0. That is,

$$|\Delta[\Delta q]\psi(x_0)|^2 = |\psi(x_0)|^2\,.$$

But, since $\Delta[\Delta q]$ is independent of x_0, we must have $|\Delta[q]|^2 = 1$ in general.

Of course, this argument is really just a suggestive motivation rather than a proof, so we take the conclusion as a postulate, rather than a proposition.

Postulate 2. *For any path $q^\mu(\lambda)$, the transition amplitude $\Delta[q]$ preserves the position probability density for the particle along the path. That is, it satisfies*

$$|\Delta[q]|^2 = 1\,. \tag{3}$$

If the configuration space for a path is expanded to be a representation of the full Poincaré group—that is, to include a matrix representation of the (homogeneous) Lorentz group as well as the four spacetime coordinates—then members of a family of "parallel" paths are related by Poincaré transformations, not just translations. This can be used as the basis for extending the spacetime path formalism to cover non-scalar particles. If, further, the assumption of flat spacetime is dropped, then it is not generally possible to construct a family of parallel paths covering all spacetime. However, one can still consider infinitesimal variations along a path corresponding to arbitrary coordinate transformations. Such further generalizations of the spacetime path approach will be explored in future papers.

The requirements of Eq. (3) and translation invariance mean that $\Delta[q]$ must have the form

$$\Delta[q] = \mathrm{e}^{\mathrm{i}S[\Delta q]}\,, \tag{4}$$

for some phase functional S. Substituting Eq. (4) into Eq. (2) gives

$$\Delta(x, x_0; [\lambda]) = \eta[\lambda] \int \mathrm{D}^4 q\, \delta^4(q(\lambda(1)) - x)\delta^4(q(\lambda(0)) - x_0)\mathrm{e}^{\mathrm{i}S[\Delta q]}\,. \tag{5}$$

So far, we have made no assumption that the particle path functions $q^\mu(\lambda)$ are differentiable. Indeed, paths under a path integral will generally *not* be differentiable. Nevertheless, it is common practice to use (with some care) path derivatives in the integrand of a path integral. This is because a path integral is defined as the limit of discretized approximations in which path derivatives are approximated as the mean value $\Delta q/\Delta\lambda$, for finite differences Δq and $\Delta\lambda$. The limit $\Delta\lambda \to 0$ is then taken over the path integral as a whole, not each

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derivative individually. Thus, even though $\lim_{\Delta\lambda\to 0}\Delta q/\Delta\lambda$ may not be defined, the path integral has a well-defined value so long as the overall path integral limit is defined. (For a discussion of some of the issues involved here, see, for example, Section 7.3 of [2]. See also the explicit example of the derivation in App. B.)

We are therefore justified in replacing the difference functions $\Delta q^\mu(\lambda)$ used in the phase functional under the path integral in Eq. (5) with the path derivatives $\dot{q}^\mu(\lambda) \equiv \mathrm{d}q^\mu/\mathrm{d}\lambda$, such that

$$\Delta q^\mu(\lambda) = \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda'\, \dot{q}^\mu(\lambda')\,,$$

letting the $q^\mu(\lambda)$ be considered as differentiable. This gives

$$\Delta(x, x_0; [\lambda]) = \eta[\lambda] \int \mathrm{D}^4 q\, \delta^4(q(\lambda(1)) - x)\delta^4(q(\lambda(0)) - x_0)\mathrm{e}^{\mathrm{i}S[\dot{q}]}\,. \tag{6}$$

which reflects the typical form of a Feynman sum over paths [2], where each path is weighted by a phase determined by the action S. Unlike the usual non-relativistic formulation, however, the path parameter here is λ, rather than time [5, 21].

Now, by dividing a path q into two paths at some arbitrary parameter value λ and propagating over each segment, we can see that

$$S[\dot{q}; \lambda_1, \lambda_0] = S[\dot{q}; \lambda_1, \lambda] + S[\dot{q}; \lambda, \lambda_0]\,, \tag{7}$$

where $S[\dot{q}; \lambda', \lambda]$ denotes the value of $S[\dot{q}]$ for the parameter range of $\dot{q}$ restricted to $[\lambda, \lambda']$. Using this property to build the total value of $S[\dot{q}]$ from infinitesimal increments leads to the following result (proved in App. A).

Proposition A (Form of the Phase Functional). *The phase functional S must have the form*

$$S[\dot{q}] = \int_{\lambda_0}^{\lambda_1} \mathrm{d}\lambda'\, L[\dot{q}; \lambda']\,, \tag{8}$$

where the parametrization domain for $\dot{q}$ is $[\lambda_0, \lambda_1]$ and $L[\dot{q}; \lambda]$ depends only on $\dot{q}$ and its higher derivatives evaluated at λ.

The question remains as to what form the function L should take. Traditionally, it is taken to be just the classical Langrangian, but, from a foundational viewpoint, one would like a better justification.

Of course, the simplest form for L would be a constant, independent of $\dot{q}$. However, this would result in a superficially divergent path integral in Eq. (6) which, when normalized,

would leave to just a trivial phase. This would not give any appropriate particle dynamics. The next simplest form for L would be for it to depend only on $\dot{q}$ and no higher derivatives. Further, since L is a scalar quantity, it must then depend only on the Lorentz-invariant scalar function

$$\dot{q}^2(\lambda) \equiv \dot{q}^\mu(\lambda)\dot{q}_\mu(\lambda) \,.$$

Taking L to further have the tractable form of a linear function of $\dot{q}^2$ gives

$$L[\dot{q}; \lambda] = L(\dot{q}^2(\lambda)) = a\dot{q}^2(\lambda) + b \,,$$

for some a and b. Now, the factor a can be fixed arbitrarily, since any variation is effectively equivalent to a reparametrization of the path parameter λ. For a free particle, it is convenient to take $a = 1/4$. If we further assume that b is always negative, we can set $b = -m^2$ and identifying m with the mass of the particle does, indeed, give a classical relativistic Lagrangian function.

As we will see in the following, evaluating the path integral in Eq. (6) with this Lagrangian function leads to the usual free-particle Feynman propagator for the particle. If, on the other hand, we take b to be positive, then the result is a similar propagator, but with an effective imaginary particle mass. Such particles are tachyons, which we will not consider further in this paper.

Postulate 3. *For a free scalar particle of mass m, the Lagrangian function is given by*

$$L(\dot{q}^2) = \frac{1}{4}\dot{q}^2 - m^2 \,. \tag{9}$$

Substituting Eq. (8) into Eq. (6) gives

$$\Delta(x, x_0; [\lambda]) = \eta[\lambda] \int \mathrm{D}^4 q \, \delta^4(q(\lambda(1)) - x)\delta^4(q(\lambda(0)) - x_0) \exp\left(\mathrm{i} \int_{\lambda(0)}^{\lambda(1)} \mathrm{d}\lambda' \, L(\dot{q}^2(\lambda')) \right) . \tag{10}$$

With the Lagrangian given by Eq. (9), it is well known that this path integral may be evaluated (see, for example, [21]). However, in the present context, some care must be taken to mathematically evaluate the integral without making any further assumptions based on field equations or underlying traditional quantum mechanics. In any case, the result (proved in App. B) is as follows.

Proposition B (Evaluation of the Path Integral). *The path integral in Eq. (10), with the Lagrangian given by Eq. (9), may be evaluated to get*

$$\Delta(x, x_0; [\lambda]) = \Delta(x - x_0; \lambda(1) - \lambda(0)) \equiv (2\pi)^{-4} \int \mathrm{d}^4 p \, \mathrm{e}^{\mathrm{i} p \cdot (x - x_0)} \mathrm{e}^{-\mathrm{i}[\lambda(1) - \lambda(0)](p^2 + m^2)} . \tag{11}$$

Note that the only dependency left of $\Delta(x - x_0; \lambda(1) - \lambda(0))$ on the parametrization $\lambda(s)$ is on the total *intrinsic path length*

$$T = \lambda(1) - \lambda(0) = \int_0^1 \mathrm{d}s\, w(s) > 0\,.$$

If we were to take $F[\lambda] = 1$ in Eq. (1) for all $\lambda(s)$, there would then be a parametrization gauge symmetry: all parametrizations that give the same intrinsic path lengths would be equivalent. Therefore, equivalent reparametrizations would be overcounted in the λ path integral of Eq. (1), so the integral would diverge.

If, on the other hand, the path integral over λ had not been included at all in Eq. (1), the result would have been to overspecify a specific path parametrization. The possible particle paths would then have been *undercounted*, missing the need to include paths of different intrinsic lengths. It is thus necessary to reduce the λ path integration in Eq. (1) to eliminate the overcounting due to the path gauge symmetry, without overspecifying the path parametrization.

In the usual fashion for a gauge symmetry, we retain the integration, but fix a specific guage. This can be easily done by including a gauge fixing delta functional in $F[\lambda]$. The gauge typically chosen is to require that $w(s) = \mathrm{d}\lambda/\mathrm{d}s$ be constant [21], which corresponds to setting

$$F[\lambda] = f(\lambda(1) - \lambda(0))\delta\left[\frac{\mathrm{d}\lambda}{\mathrm{d}s} - [\lambda(1) - \lambda(0)]\right]\,,$$

for some real function $f(T)$. Using this in Eq. (1) gives

$$\Delta(x, x_0) = \int_0^\infty \mathrm{d}T\, f(T)\Delta(x - x_0; T)\,.$$

In the following, we will generally assume equal weighting of all parametrizations, that is $f(T) = 1$. However, in Sec. III D, we will see that an alternate choice provides a fruitful path for regularizing the infinite integrals that appear in the formalism for interacting particles. Nevertheless, assuming, for now, that $f(T) = 1$, gives

$$\Delta(x, x_0) = \int_0^\infty \mathrm{d}T\, \Delta(x - x_0; T) = (2\pi)^{-4} \int \mathrm{d}^4 p\, \mathrm{e}^{\mathrm{i}p \cdot (x - x_0)} \int_0^\infty \mathrm{d}T\, \mathrm{e}^{-\mathrm{i}T(p^2 + m^2)}\,. \qquad (12)$$

This can be evaluated by introducing a convergence factor $\exp(-T\varepsilon)$, for infinitesimal ε, resulting in just the Feynman propagator

$$\Delta(x, x_0) = \Delta(x - x_0) \equiv -\mathrm{i}(2\pi)^{-4} \int \mathrm{d}^4 p\, \frac{\mathrm{e}^{\mathrm{i}p \cdot (x - x_0)}}{p^2 + m^2 - \mathrm{i}\varepsilon}\,.$$

The integration of T from 0 to ∞ in Eq. (12) is similar to the integration carried out by Nambu [9], based on previous work of Fock [8], in order to obtain the Feynman propagator. Note, though, that this integration arises naturally here as the guage-fixed reduction of the path parametrization integral in Eq. (1).

The relationship between the propagator $\Delta(x - x_0)$ and $\Delta(x - x_0; T)$ can be viewed in another way, which will also prove useful in Sec. III D. For $T > 0$,

$$
\begin{aligned}
\Delta(x - x_0; T) &= e^{-iTm^2} \int d^4p \, e^{ip \cdot (x - x_0)} \int_0^\infty dT' \, e^{-iT'p^2} \delta(T' - T) \\
&= (2\pi)^{-1} e^{-iTm^2} \int d^4p \, e^{ip \cdot (x - x_0)} \int_0^\infty dT' \, e^{-iT'p^2} \int dm'^2 \, e^{-i(T' - T)m'^2} \qquad (13) \\
&= (2\pi)^{-1} e^{-iTm^2} \int dm'^2 \, e^{iTm'^2} \Delta(x - x_0; m'^2) ,
\end{aligned}
$$

where

$$
\Delta(x - x_0; m'^2) \equiv \int_0^\infty dT' \int d^4p \, e^{ip \cdot (x - x_0)} e^{-iT'(p^2 + m'^2)} = -i(2\pi)^{-4} \int d^4p \, \frac{e^{ip \cdot (x - x_0)}}{p^2 + m'^2 - i\varepsilon} . \qquad (14)
$$

This form for $\Delta(x - x_0; T)$ is essentially that of the parametrized Green's function derived by Horwitz et al. for parametrized quantum field theory [37, 38] as a superposition of propagators for different mass states (see also [39, 40]). Equation (13) differs from those references in the factor $\exp(-iTm^2)$. As a result of this factor, integrating Eq. (13) over T as in Eq. (12) effectively acts as a Fourier transform, resulting in a propagator with mass sharply defined at m.

B. Free Particle Position States

The path integral form for $\Delta(x - x_0; \lambda - \lambda_0)$ given in Eq. (2) is essentially the same as that of the path integral for the non-relativistic *kernel* [2], except that λ is used as the evolution parameter instead of t. Therefore, $\Delta(x - x_0; \lambda - \lambda_0)$ has similar properties as a propagation kernel in λ:

$$
\int d^4x_1 \, \Delta(x - x_1; \lambda - \lambda_1) \Delta(x_1 - x_0; \lambda_1 - \lambda_0) = \Delta(x - x_0; \lambda - \lambda_0)
$$

and

$$
\Delta(x - x_0; \lambda - \lambda_0)^* = \Delta(x_0 - x; \lambda_0 - \lambda) .
$$

Given these properties, define a *family* of probability amplitude functions $\psi(x; \lambda)$, for which

$$\psi(x; \lambda) = \int d^4x_0 \, \Delta(x - x_0; \lambda - \lambda_0)\psi(x_0; \lambda_0) \,, \tag{15}$$

for any λ and λ_0, normalized such that

$$\int d^4x \, |\psi(x; \lambda)|^2 = 1 \,, \tag{16}$$

for each λ. Formally, these functions are probability amplitudes for the position x, with λ serving as an index identifying individual functions in the family. However, they can be interpreted as just the parametrized probability amplitude functions defined by Stueckelberg [6]. In this sense, the $\psi(x; \lambda)$ represent the probability amplitude for a particle to reach position x at the point along its path with parameter value λ.

Note that

$$i\frac{\partial}{\partial \lambda}\Delta(x - x_0; \lambda - \lambda_0) = (2\pi)^{-4} \int d^4p \, e^{ip\cdot(x-x_0)}(p^2 + m^2)e^{-i(\lambda-\lambda_0)(p^2+m^2)} \,.$$

This means that $\psi(x; \lambda)$, as given by Eq. (15), satisfies

$$-i\frac{\partial}{\partial \lambda}\psi(x; \lambda) = \left(\frac{\partial^2}{\partial x^2} - m^2\right)\psi(x; \lambda) \,. \tag{17}$$

Equation (17) is a generalized Schrödinger equation, such as proposed by Stueckelberg [7]. However, Stueckelberg and subsequent authors [4, 12, 15] used a Hamiltonian of the form $(2m)^{-1}\partial^2/\partial x^2$, by analogy with non-relativistic mechanics, rather than the form of Eq. (17) (though [41] uses a Hamiltonian form similar to Eq. (17)). This difference is the origin of the extra factor $\exp(-iTm^2)$ in Eq. (13) relative to [37, 38].

The properties of the kernel $\Delta(x - x_0; \lambda - \lambda_0)$ also allow for the definition of a consistent family of *position state* bases $|x; \lambda\rangle$, such that

$$\psi(x; \lambda) = \langle x; \lambda|\psi\rangle \,, \tag{18}$$

given a single Hilbert space state vector $|\psi\rangle$. These position states are normalized such that

$$\langle x'; \lambda|x; \lambda\rangle = \delta^4(x' - x) \,.$$

for each value of λ. Further, it follows from Eqs. (15) and (18) that

$$\Delta(x - x_0; \lambda - \lambda_0) = \langle x; \lambda|x_0; \lambda_0\rangle \,. \tag{19}$$

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Thus, $\Delta(x - x_0; \lambda - \lambda_0)$ effectively defines a unitary transformation between the various Hilbert space bases $|x; \lambda\rangle$, indexed by the parameter λ.

Finally, the overall state for propagation from x_0 to x is given by the superposition of the states for paths of all intrinsic lengths. If we fix $q^\mu(\lambda_0) = x_0^\mu$, then $|x; \lambda\rangle$ already includes all paths of length $\lambda - \lambda_0$. Therefore, the overall state $|x\rangle$ for the particle to arrive at x should be given by the superposition of the states $|x; \lambda\rangle$ for all $\lambda > \lambda_0$:

$$|x\rangle \equiv \int_{\lambda_0}^\infty \mathrm{d}\lambda \, |x; \lambda\rangle .\tag{20}$$

Then, using Eq. (19),

$$\langle x|x_0; \lambda_0\rangle = \int_{\lambda_0}^\infty \mathrm{d}\lambda \, \Delta(x - x_0; \lambda - \lambda_0) = \int_0^\infty \mathrm{d}\lambda \, \Delta(x - x_0; \lambda) = \Delta(x - x_0) .\tag{21}$$

Now, the $|x\rangle$ are not actually proper Hilbert space states, since $\langle x|x_0\rangle$ is infinite (as can be see by integrating Eq. (21) over λ_0). Nevertheless, via Eq. (12), the corresponding bras $\langle x|$ can be considered to be well-defined functions on proper, normalizable states $|\psi\rangle$ such that

$$\langle x|\psi\rangle = \int \mathrm{d}^4 x_0 \, \Delta(x - x_0)\psi(x_0; \lambda_0)$$

is the transition amplitude for a particle with known probability amplitude $\psi(x_0; \lambda_0)$ at λ_0 to eventually reach position x at some $\lambda > \lambda_0$. We will thus continue to use $|x\rangle$ as a formal quantity, with the understanding that it is really just a shorthand for constructing propagators and transition amplitudes.

C. On-Shell Particle and Antiparticle States

The states constructed so far have naturally been off-shell states. That is, they represent what are normally considered to be "virtual" particles. However, rather than simply imposing the on-shell mass condition to obtain "physical" states, on-shell states will be constructed in this subsection as the infinite time limit of off-shell states. That is, particles with paths that, in the limit, are unbounded in time will turn out to be naturally on-shell.

In order to take a time limit, it is necessary to make some distinction between past and future that can be used as the basis for taking the limit. For this purpose, divide the set of all possible paths q^μ that end at some specific $q^\mu(\lambda) = x^\mu$ into two subsets: those that begin (at $q^\mu(\lambda_0) = x_0^\mu$) in the *past* of x and those that begin in the *future* of x.

Outside of the light cone of x, the division into future and past is, of course, not Lorentz covariant and depends on the choice of a specific coordinate system. However, when we take the time limit, the light cone expands to cover all space, and, in this limit, the division into particle and antiparticle becomes fully coordinate system independent. The possibility of the particle/antiparticle distinction being coordinate system dependent in anything other than the infinite time limit is a subject for future exploration.

Now, particles are normally considered to propagate *from* the past *to* the future. On the other hand, *antiparticles* may be considered to propagate from the *future* into the *past* [3, 6, 7].

Postulate 4. *Normal particle states $|x_+\rangle$ are such that*

$$\langle x_+|x_0; \lambda_0\rangle = \theta(x^0 - x_0^0)\Delta(x - x_0)\,.$$

Antiparticle states $|x_-\rangle$ are such that

$$\langle x_-|x_0; \lambda_0\rangle = \theta(x_0^0 - x^0)\Delta(x - x_0)\,.$$

Using the usual decomposition of the Feynman propagator (see, for example, Section 6.2 of [42])

$$\Delta(x - x_0) = \theta(x^0 - x_0^0)\Delta_+(x - x_0) + \theta(x_0^0 - x^0)\Delta_-(x - x_0)\,, \tag{22}$$

where

$$\Delta_\pm(x - x_0) \equiv (2\pi)^{-3}\int d^3 p\, \frac{e^{i[\mp\omega_{\bm{p}}(x^0 - x_0^0) + \bm{p}\cdot(\bm{x} - \bm{x}_0)]}}{2\omega_{\bm{p}}}\,, \tag{23}$$

with $\omega_{\bm{p}} \equiv \sqrt{\bm{p}^2 + m^2}$, it is clear that

$$\langle x_\pm|x_0; \lambda_0\rangle = \theta(\pm(x^0 - x_0^0))\Delta(x - x_0) = \theta(\pm(x^0 - x_0^0))\Delta_\pm(x - x_0)\,. \tag{24}$$

We would now like to take the time limits for future and past directed particle and antiparticle states. In doing this, one cannot expect to hold the 3-position of the path end point constant. However, for a free particle, it is reasonable to take the particle *3-momentum* as being fixed. Therefore, consider the state of a particle or antiparticle with a 3-momentum $\bm{p}$ at a certain time t. (The importance of the specific factor $\exp(\mp i\omega_{\bm{p}}t)$ in the definition below will become clear in a moment.)

Postulate 5. *The state of a particle (+) or antiparticle (−) with 3-momentum $\boldsymbol{p}$ is given by*

$$
\begin{aligned}
|t, \boldsymbol{p}_{\pm}\rangle &\equiv (2\pi)^{-3/2} \int \mathrm{d}^3 x \, \mathrm{e}^{\mathrm{i}(\mp\omega_{\boldsymbol{p}}t + \boldsymbol{p}\cdot\boldsymbol{x})} |t, \boldsymbol{x}_{\pm}\rangle \\
&= (2\pi)^{-1/2} \mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t} \int \mathrm{d}p^0 \, \mathrm{e}^{\mathrm{i}p^0 t} |p_{\pm}\rangle \,,
\end{aligned}
\tag{25}
$$

where

$$
|p_{\pm}\rangle \equiv (2\pi)^{-2} \int \mathrm{d}^4 x \, \mathrm{e}^{\mathrm{i}p\cdot x} |x_{\pm}\rangle
\tag{26}
$$

is the corresponding 4-momentum state.

Let

$$
\begin{aligned}
|t_0, \boldsymbol{p}_{\pm}; \lambda_0\rangle &\equiv (2\pi)^{-3/2} \int \mathrm{d}^3 x \, \mathrm{e}^{\mathrm{i}(\mp\omega_{\boldsymbol{p}}t_0 + \boldsymbol{p}\cdot\boldsymbol{x})} |t_0, \boldsymbol{x}; \lambda_0\rangle \\
&= (2\pi)^{-1/2} \mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t_0} \int \mathrm{d}p^0 \, \mathrm{e}^{\mathrm{i}p^0 t_0} |p; \lambda_0\rangle \,,
\end{aligned}
\tag{27}
$$

where

$$
|p; \lambda_0\rangle \equiv (2\pi)^{-2} \int \mathrm{d}^4 x \, \mathrm{e}^{\mathrm{i}p\cdot x} |x; \lambda_0\rangle \,.
\tag{28}
$$

Substituting from Eqs. (26), (28) and (24),

$$
\begin{aligned}
\langle p_{\pm}|p_0; \lambda_0\rangle &= (2\pi)^{-4} \int \mathrm{d}^4 x \, \mathrm{d}^4 x_0 \, \mathrm{e}^{-\mathrm{i}p\cdot x} \mathrm{e}^{\mathrm{i}p_0\cdot x_0} \theta(\pm(x^0 - x_0^0))\Delta_{\pm}(x - x_0) \\
&= \delta^4(p - p_0)\Delta_{\pm}(p) \,,
\end{aligned}
\tag{29}
$$

where

$$
\Delta_{\pm}(p) \equiv \int \mathrm{d}^4 x \, \mathrm{e}^{-\mathrm{i}p\cdot x} \theta(\pm x^0)\Delta_{\pm}(x) \,.
\tag{30}
$$

Substituting Eq. (23) into Eq. (30) gives

$$
\begin{aligned}
\Delta_{\pm}(p) &= \int \mathrm{d}^4 x \, \mathrm{e}^{-\mathrm{i}p\cdot x} \theta(\pm x^0)(2\pi)^{-3} \int \mathrm{d}^3 p' \, (2\omega_{\boldsymbol{p}'})^{-1} \mathrm{e}^{\mathrm{i}(\mp\omega_{\boldsymbol{p}'}x^0 + \boldsymbol{p}'\cdot\boldsymbol{x})} \\
&= \int \mathrm{d}t \, \mathrm{e}^{\mathrm{i}p^0 t} \theta(\pm t) \int \mathrm{d}^3 p' \, (2\omega_{\boldsymbol{p}'})^{-1} \mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}'}t}(2\pi)^{-3} \int \mathrm{d}^3 x \, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}' - \boldsymbol{p})\cdot\boldsymbol{x}} \\
&= \int \mathrm{d}t \, \mathrm{e}^{\mathrm{i}p^0 t} \theta(\pm t) \int \mathrm{d}^3 p' \, (2\omega_{\boldsymbol{p}'})^{-1} \mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}'}t} \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \\
&= (2\omega_{\boldsymbol{p}})^{-1} \int \mathrm{d}t \, \theta(\pm t)\mathrm{e}^{\mathrm{i}(p^0 \mp \omega_{\boldsymbol{p}})t} \,.
\end{aligned}
\tag{31}
$$

Using Eq. (29) (and the completeness of the $|p; \lambda_0\rangle$ states) in Eq. (25), and substituting from Eq. (31) for $\Delta_{\pm}(p)$, then gives

$$
\begin{aligned}
|t, \boldsymbol{p}_{\pm}\rangle &= (2\pi)^{-1/2} \mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t} \int \mathrm{d}p^0 \, \mathrm{e}^{\mathrm{i}p^0 t} \Delta_{\pm}(p)^* |p; \lambda_0\rangle \\
&= (2\pi)^{-1/2}(2\omega_{\boldsymbol{p}})^{-1} \int \mathrm{d}t' \, \theta(\pm t')\mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}(t - t')} \int \mathrm{d}p^0 \, \mathrm{e}^{\mathrm{i}p^0(t - t')} |p; \lambda_0\rangle \,.
\end{aligned}
$$

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Change variables $t' \to t - t_0$ to get

$$|t, \boldsymbol{p}_\pm\rangle = (2\pi)^{-1/2}(2\omega_{\boldsymbol{p}})^{-1} \int dt_0\, \theta(\pm(t - t_0)) \int dp^0\, e^{i(p^0 \mp \omega_{\boldsymbol{p}})t_0}|p; \lambda_0\rangle$$

$$= \begin{cases} (2\omega_{\boldsymbol{p}})^{-1} \int_{-\infty}^{t} dt_0\, |t_0, \boldsymbol{p}_+; \lambda_0\rangle\,, \\[2mm] (2\omega_{\boldsymbol{p}})^{-1} \int_{t}^{+\infty} dt_0\, |t_0, \boldsymbol{p}_-; \lambda_0\rangle\,. \end{cases} \tag{32}$$

It is then straightforward to take the time limit $t \to \pm\infty$. Note that

$$\int_{-\infty}^{+\infty} dt_0\, |t_0, \boldsymbol{p}_\pm; \lambda_0\rangle = (2\pi)^{-1/2} \int dp^0 \int dt_0\, e^{i(p^0 \mp \omega_{\boldsymbol{p}})t_0}|p; \lambda_0\rangle$$

$$= (2\pi)^{-1/2} \int dp^0\, (2\pi)\delta(p^0 \mp \omega_{\boldsymbol{p}})|p; \lambda_0\rangle$$

$$= (2\pi)^{1/2}|\pm\omega_{\boldsymbol{p}}, \boldsymbol{p}; \lambda_0\rangle\,.$$

Therefore

$$|\boldsymbol{p}_\pm\rangle \equiv \lim_{t \to \pm\infty} |t, \boldsymbol{p}_\pm\rangle = (2\pi)^{1/2}(2\omega_{\boldsymbol{p}})^{-1}|\pm\omega_{\boldsymbol{p}}, \boldsymbol{p}; \lambda_0\rangle\,. \tag{33}$$

Thus, a normal particle $(+)$ or antiparticle $(-)$ that has 3-momentum $\boldsymbol{p}$ as $t \to \pm\infty$ is *on-shell*, with energy $\pm\omega_{\boldsymbol{p}}$. Such on-shell particles are unambiguously normal particles or antiparticles, independent of choice of coordinate system. (Note that these states are similar to the "mass representation" states of [12].)

Note also that the factor of $\exp(\mp i\omega_{\boldsymbol{p}}t)$ in the definition of $|t, \boldsymbol{p}_\pm\rangle$ (Eq. (25)) is not arbitrary. Without this, a factor of $\exp(\pm i\omega_{\boldsymbol{p}}t)$ would remain in Eq. (32), making it impossible to take the limit $t \to \pm\infty$.

D. On-Shell Probability Interpretation

Unfortunately, the states defined in Eq. (33) are not normalizable using the usual inner product, since

$$\langle \boldsymbol{p}'_\pm|\boldsymbol{p}_\pm\rangle = 2\pi(2\omega_{\boldsymbol{p}})^{-2}\delta(0)\delta^3(\boldsymbol{p}' - \boldsymbol{p})$$

is infinite. In [12], this is handled by allowing the mass m to vary, even though the energy is fixed at $\sqrt{\boldsymbol{p}^2 + m^2}$. Here we will take a different approach, noting that, from Eq. (28),

$$\langle p'; \lambda|p; \lambda\rangle = \delta^4(p' - p)\,.$$

Using this and Eq. (33), we clearly have

$$\langle \boldsymbol{p}_\pm|p_0; \lambda_0\rangle = (2\pi)^{1/2}(2\omega_{\boldsymbol{p}})^{-1}\delta(\pm\omega_{\boldsymbol{p}} - p_0^0)\delta^3(\boldsymbol{p} - \boldsymbol{p}_0)\,. \tag{34}$$

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Moreover, from this equation and Eq. (27),

$$\langle \boldsymbol{p}_\pm | t_0, \boldsymbol{p}_{0\pm}; \lambda_0 \rangle = (2\pi)^{-1/2} \mathrm{e}^{\mp i\omega_{\boldsymbol{p}} t_0} \int \mathrm{d}p_0^0 \, \mathrm{e}^{ip_0^0 t_0} (2\pi)^{1/2} (2\omega_{\boldsymbol{p}})^{-1} \delta(\pm\omega_{\boldsymbol{p}} - p_0^0) \delta^3(\boldsymbol{p} - \boldsymbol{p}_0)$$
$$= (2\omega_{\boldsymbol{p}})^{-1} \delta^3(\boldsymbol{p} - \boldsymbol{p}_0) \,, \tag{35}$$

for any value of t_0. This is essentially the basis for an "induced" inner product, in the sense of [23, 43].

Let $\mathcal{H}$ be the Hilbert space of the $|x; \lambda_0\rangle$ and let $\mathcal{H}_t$ be the subspaces spanned by the $|t, \boldsymbol{x}; \lambda_0\rangle$, for each t, forming a foliation of $\mathcal{H}$. Now, from Eq. (27), it is clear that the particle and antiparticle 3-momentum states $|t, \boldsymbol{p}_\pm; \lambda_0\rangle$ also each span $\mathcal{H}_t$. In these representations, states in $\mathcal{H}_t$ have the form

$$|t, \psi_\pm; \lambda_0\rangle = \int \mathrm{d}^3 p \, \psi(\boldsymbol{p}) |t, \boldsymbol{p}_\pm; \lambda_0\rangle \,, \tag{36}$$

for square-integrable functions $\psi(\boldsymbol{p})$. Conversely, it follows from Eq. (35) that a probability amplitude $\psi(\boldsymbol{p})$ is given by

$$\psi(\boldsymbol{p}) = (2\omega_{\boldsymbol{p}}) \langle \boldsymbol{p}_\pm | t, \psi_\pm; \lambda_0 \rangle \,. \tag{37}$$

Let $\mathcal{H}'_t$ be the space of linear functions dual to $\mathcal{H}_t$. Via Eq. (37), the bra states $\langle \boldsymbol{p}_\pm |$ can be considered to be members of $\mathcal{H}'_t$, for all t. Indeed, they span two common subspaces $\mathcal{H}'_\pm$ of the $\mathcal{H}'_t$, the states of which have the form

$$\langle \psi_\pm | = \int \mathrm{d}^3 p \, \psi(\boldsymbol{p})^* \langle \boldsymbol{p}_\pm | \,.$$

Now, define an inner product on the functions $\psi(\boldsymbol{p})$ such that

$$(\psi_1, \psi_2) \equiv \langle \psi_{1\pm} | t, \psi_{2\pm}; \lambda_0 \rangle = \int \frac{\mathrm{d}^3 p}{2\omega_{\boldsymbol{p}}} \psi_1(\boldsymbol{p})^* \psi_2(\boldsymbol{p}) \,, \tag{38}$$

where the second equality follows from Eq. (35). Equipped with this inner product, each $\mathcal{H}_t$ is itself a Hilbert space of the wave functions $\psi(\boldsymbol{p})$. Note that it is the states of the *dual* spaces $\mathcal{H}'_\pm$ that naturally satisfy the on-shell constraint $\langle \psi_\pm | \hat{H} = 0$ (as suggested by, for example, [44]).

The operators $(2\omega_{\boldsymbol{p}}) |t_0, \boldsymbol{p}_\pm; \lambda_0 \rangle \langle \boldsymbol{p}_\pm |$ are self-adjoint under the inner product given in Eq. (38), in the sense that the conjugate of

$$(2\omega_{\boldsymbol{p}}) \langle \psi_\pm | t_0, \boldsymbol{p}_\pm; \lambda_0 \rangle \langle \boldsymbol{p}_\pm | = \psi(\boldsymbol{p})^* \langle \boldsymbol{p}_\pm |$$

is

$$(2\omega_{\boldsymbol{p}})|t_0, \boldsymbol{p}_{\pm}; \lambda_0\rangle\langle \boldsymbol{p}_{\pm}|t, \psi_{\pm}; \lambda_0\rangle = |t_0, \boldsymbol{p}_{\pm}; \lambda_0\rangle\psi(\boldsymbol{p})$$

for that inner product. Further,

$$\int \mathrm{d}^3p \, (2\omega_{\boldsymbol{p}})\langle\psi_{1\pm}|t_0, \boldsymbol{p}_{\pm}; \lambda_0\rangle\langle \boldsymbol{p}_{\pm}|t, \psi_{2\pm}; \lambda_0\rangle = \int \frac{\mathrm{d}^3p}{2\omega_{\boldsymbol{p}}}\psi_1(\boldsymbol{p})^*\psi_2(\boldsymbol{p}) = \langle\psi_{1\pm}|t, \psi_{2\pm}; \lambda_0\rangle \,,$$

which gives the effective resolution of the identity

$$\int \mathrm{d}^3p \, (2\omega_{\boldsymbol{p}})|t_0, \boldsymbol{p}_{\pm}; \lambda_0\rangle\langle \boldsymbol{p}_{\pm}| = 1 \,. \tag{39}$$

In fact, such a resolution of the identity generally holds for families of conjugate bra and ket states with a *bi-orthonormality relationship* such as Eq. (35) (see [45] and App. A.8.1 of [46]). We can, therefore, take the operator $(2\omega_{\boldsymbol{p}})|t_0, \boldsymbol{p}_{\pm}; \lambda_0\rangle\langle \boldsymbol{p}_{\pm}|$ to represent the quantum proposition that an *on-shell* particle or antiparticle has the *3-momentum* $\boldsymbol{p}$. Then, with the normalization

$$(\psi, \psi) = \int \frac{\mathrm{d}^3p}{2\omega_{\boldsymbol{p}}}|\psi(\boldsymbol{p})|^2 = 1 \,,$$

$|\psi(\boldsymbol{p})|^2$ is the corresponding probability density in 3-momentum space.

Finally, consider that $|t, \boldsymbol{x}; \lambda_0\rangle$ is an eigenstate of the 3-position operator $\hat{\boldsymbol{X}}$, representing a particle localized at the 3-position $\boldsymbol{x}$ at time t. From Eq. (37), and using the inverse Fourier transform of Eq. (28) with Eq. (33), its 3-momentum wave function in $\mathcal{H}_t$ is

$$(2\omega_{\boldsymbol{p}})\langle \boldsymbol{p}_{\pm}|t, \boldsymbol{x}; \lambda_0\rangle = (2\pi)^{-3/2}\mathrm{e}^{\mathrm{i}(\pm\omega_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})} \,. \tag{40}$$

This is just a plane wave, and it is an eigenfunction of the operator

$$\mathrm{e}^{\pm\mathrm{i}\omega_{\boldsymbol{p}}t}\,\mathrm{i}\frac{\partial}{\partial\boldsymbol{p}}\mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t} \,,$$

which is the traditional momentum representation $\mathrm{i}\partial/\partial\boldsymbol{p}$ of the 3-position operator $\hat{\boldsymbol{X}}$, translated to time t.

This result contrasts with the well-known result of Newton and Wigner [47], who conclude that a localized particle wave function satisfying the Klein-Gordon equation is an eigenfunction of

$$\mathrm{i}\left(\frac{\partial}{\partial\boldsymbol{p}} - \frac{\boldsymbol{p}}{2\omega_{\boldsymbol{p}}^2}\right) \,,$$

which has an extra term over the expected $\mathrm{i}\partial/\partial\boldsymbol{p}$. The key reason for this difference is our use of the 3-momentum basis $|t, \boldsymbol{p}_{\pm}; \lambda_0\rangle$. With the dual basis $(2\omega_{\boldsymbol{p}})\langle \boldsymbol{p}_{\pm}|$ from Eq. (35), this leads to the relation given in Eq. (37) and used in Eq. (40).

In contrast, the traditional formalism assumes that both bra and ket states are on-shell. Instead of the time-dependent spaces $\mathcal{H}_t$, the spaces $\mathcal{H}_\pm$ are used, with on-shell ket basis states $|\boldsymbol{p}_\pm\rangle$ that are dual to the bra states $\langle \boldsymbol{p}_\pm|$ under an inner product such that, instead of Eq. (35), one has

$$(2\omega_{\boldsymbol{p}'})^{1/2}\langle \boldsymbol{p}'_\pm|\boldsymbol{p}_\pm\rangle (2\omega_{\boldsymbol{p}})^{1/2} = \delta^3(\boldsymbol{p}' - \boldsymbol{p}),$$

where the factor of $(2\omega_{\boldsymbol{p}})^{1/2}$ is introduced symmetrically on dual bra and ket states in order provide an orthonormal basis. If we were to use the traditional dual basis $(2\omega_{\boldsymbol{p}})^{1/2}\langle \boldsymbol{p}_\pm|$, instead of $(2\omega_{\boldsymbol{p}})\langle \boldsymbol{p}_\pm|$, the wave function of $|t, \boldsymbol{x}; \lambda_0\rangle$ would be

$$(2\omega_{\boldsymbol{p}})^{1/2}\langle \boldsymbol{p}_\pm|t, \boldsymbol{x}; \lambda_0\rangle = (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1/2}\mathrm{e}^{\mathrm{i}(\pm\omega_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})}. \tag{41}$$

At $t = 0$ this is exactly the Newton-Wigner wave function for a localized particle [47].

Note that Eq. (40) is effectively related to Eq. (41) by a scalar Foldy-Wouthuysen transformation [48, 49]. This makes sense, since the Foldy-Wouthuysen transformation produces a representation that separates positive and negative energy states (particles and antiparticles) and gives a reasonable non-relativistic limit.

Indeed, from Eq. (27) we can easily see that the time evolution of the 3-momentum states $|t, \boldsymbol{p}_\pm; \lambda_0\rangle$ is given by

$$\mathrm{e}^{\mathrm{i}\hat{P}^0\Delta t}|t, \boldsymbol{p}_\pm; \lambda_0\rangle = \mathrm{e}^{\pm\mathrm{i}\omega_{\boldsymbol{p}}\Delta t}|t + \Delta t, \boldsymbol{p}_\pm; \lambda_0\rangle = \mathrm{e}^{\pm\mathrm{i}\hat{H}_{FW}\Delta t}|t + \Delta t, \boldsymbol{p}_\pm; \lambda_0\rangle,$$

where

$$\hat{H}_{FW} = (\hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{P}} + m^2)^{1/2}$$

is the scalar Foldy-Wouthuysen Hamiltonian and the $\hat{P}^\mu$ are the generators of spacetime translations. Define the operation of time translation on the time-dependent states $|t, \psi; \lambda_0\rangle$ so that

$$|t + \Delta t, \psi_\pm; \lambda_0\rangle = \mathrm{e}^{\mathrm{i}\hat{P}^0\Delta t}|t, \psi_\pm; \lambda_0\rangle.$$

Substituting Eq. (36) then gives

$$|t + \Delta t, \psi_\pm; \lambda_0\rangle = \int \mathrm{d}^3 p\, \psi(t, \boldsymbol{p})\mathrm{e}^{\mathrm{i}\hat{P}^0\Delta t}|t, \boldsymbol{p}_\pm; \lambda_0\rangle$$

$$= \int \mathrm{d}^3 p\, \psi(t, \boldsymbol{p})\mathrm{e}^{\pm\mathrm{i}\omega_{\boldsymbol{p}}\Delta t}|t + \Delta t, \boldsymbol{p}_\pm; \lambda_0\rangle$$

$$= \int \mathrm{d}^3 p\, \psi(t + \Delta t, \boldsymbol{p})|t + \Delta t, \boldsymbol{p}_\pm; \lambda_0\rangle,$$

where the time-dependence of the 3-momentum wave function has been made explicit, with time evolution given by

$$\psi(t + \Delta t, \boldsymbol{p}) = e^{\pm i\omega_{\boldsymbol{p}}\Delta t}\psi(t, \boldsymbol{p}) \,.$$

In the non-relativistic limit, for positive-energy particles, $\omega_{\boldsymbol{p}} \approx m + \boldsymbol{p}^2/2m$, and this time evolution reduces to time evolution according to the usual non-relativistic Hamiltonian (up to the momentum-independent phase factor $\exp(im\Delta t)$).

E. Free Multiparticle States

The formalism introduced in the previous sections can be extended in a straightforward way to a Fock space of non-interacting multiparticle states. In order to allow for multiparticle states with different types of particles, extend the position state of each individual particle with a *particle type index n*, such that

$$\langle x', n'; \lambda | x, n; \lambda \rangle = \delta_n^{n'} \delta^4(x' - x) \,.$$

Then, construct a basis for the Fock space of multiparticle states as symmetrized products of N single particle states:

$$|x_1, n_1, \lambda_1; \ldots; x_N, n_N, \lambda_N\rangle \equiv (N!)^{-1/2} \sum_{\text{perms } \mathcal{P}} |x_{\mathcal{P}1}, n_{\mathcal{P}1}; \lambda_{\mathcal{P}1}\rangle \cdots |x_{\mathcal{P}N}, n_{\mathcal{P}N}; \lambda_{\mathcal{P}N}\rangle, \qquad (42)$$

where the sum is over all permutations $\mathcal{P}$ of $1, 2, \ldots, N$. (Since only scalar particles are being considered in the present work, only Bose-Einstein statistics need be accounted for.)

Define multiparticle states $|x_1, n_1; \ldots; x_N, n_N\rangle$ as similarly symmetrized products of $|x\rangle$ states. Then,

$$\langle x'_1, n'_1; \ldots; x'_N, n'_N | x_1, n_1, \lambda_0; \ldots; x_N, n_N, \lambda_0 \rangle = \sum_{\text{perms } \mathcal{P}} \prod_{i=1}^{N} \delta_{n_i}^{n'_{\mathcal{P}i}} \Delta(x'_{\mathcal{P}i} - x_i; m_i^2), \qquad (43)$$

where m_i is the mass of particles of type n_i. Note that the use of the same parameter value λ_0 for the starting point of each particle path is simply a matter of convenience, using the path parameterization guage freedom to choose this value arbitrarily. The intrinsic lengths of each particle path are still integrated over *separately* in $|x_1, n_1; \ldots; x_N, n_N\rangle$, which is important for obtaining the proper particle propagator factors in Eq. (43). Nevertheless, by using λ_0

as a common starting parameter value, we can make the small notational simplification of not repeating it multiple times in $|x_1, n_1, \lambda_0; \ldots; x_N, n_N, \lambda_0\rangle$, defining, instead,

$$|x_1, n_1; \ldots; x_N, n_N; \lambda_0\rangle \equiv |x_1, n_1, \lambda_0; \ldots; x_N, n_N, \lambda_0\rangle .$$

Following the same procedures as in Sec. II C for each particle in a multiparticle state, it is straightforward to construct the multiparticle three momentum states $|t_1, \boldsymbol{p}_{1\pm}, n_1; \ldots; t_N, \boldsymbol{p}_{N\pm}, n_N\rangle$ and $|t_1, \boldsymbol{p}_{1\pm}, n_1; \ldots; t_N, \boldsymbol{p}_{N\pm}, n_N; \lambda_0\rangle$. Note that each particle may be *either* a normal particle $(+)$ or an antiparticle $(-)$. Then, to obtain on-shell states we need to take $t_i \to +\infty$ in $|t_1, \boldsymbol{p}_{1\pm}, n_1; \ldots; t_N, \boldsymbol{p}_{N\pm}, n_N\rangle$ for particles, but $t_i \to -\infty$ for antiparticles. This results in the multiparticle on-shell states $|\boldsymbol{p}_{1\pm}, n_1; \ldots; \boldsymbol{p}_{N\pm}, n_N\rangle$.

Now, it can be seen that the $|\boldsymbol{p}_{1\pm}, n_1; \ldots; \boldsymbol{p}_{N\pm}, n_N\rangle$ states may not always be particularly convenient, since they describe normal particles at $t = +\infty$ and antiparticles at $t = -\infty$. For describing the asymptotic state of outgoing particles from a scattering process, for instance, we would like to take the limit for all particles and antiparticles together as $t \to +\infty$.

To do this, we can take the viewpoint of considering antiparticles to be positive energy particles traveling forwards in time, rather than negative energy particles traveling backwards in time. Since both particles and their antiparticles will then have positive energy, it becomes necessary to explicitly label antiparticles with separate (though related) types from their corresponding particles. Let n_+ denote the type label for a normal particle type and n_- denote the corresponding antiparticle type.

For normal particles of type n_+, position states are defined as in Eq. (24):

$$\langle x, n_+ | x_0, n_+; \lambda_0\rangle = \theta(x^0 - x_0^0)\Delta_+(x - x_0) .$$

For antiparticles of type n_-, however, position states are now defined such that

$$\langle x, n_- | x_0, n_-; \lambda_0\rangle = \theta(x^0 - x_0^0)\Delta_-(x_0 - x) . \tag{44}$$

Note the reversal with respect to Eq. (24) of x_0 and x on the righthand side of this equation.

Using Eq. (23), the Fourier transform of Eq. (44) is

$$\int \mathrm{d}^4 x \, \mathrm{e}^{-\mathrm{i}p \cdot x}\theta(x^0)\Delta_-(-x) = \int \mathrm{d}^4 x \, \mathrm{e}^{-\mathrm{i}p \cdot x}\theta(x^0)\Delta_+(x^0, -\boldsymbol{x})$$

$$= \int \mathrm{d}^4 x \, \mathrm{e}^{\mathrm{i}(p^0 x^0 + \boldsymbol{p} \cdot \boldsymbol{x})}\theta(x^0)\Delta_+(x)$$

$$= \Delta_+(p^0, -\boldsymbol{p}) ,$$

where $\Delta_+(p)$ is as given in Eq. (30). From this we can see that carrying through the derivation for antiparticle 3-momentum states will, indeed, give positive energy states, but with reversed three momentum:

$$|t, \boldsymbol{p}, n_-\rangle = (2\omega_{\boldsymbol{p}})^{-1} \int_{-\infty}^{t} \mathrm{d}t_0 \, |t_0, \boldsymbol{p}, n_-; \lambda_0\rangle \,,$$

where

$$|t_0, \boldsymbol{p}, n_-; \lambda_0\rangle = |t_0, -\boldsymbol{p}_+, n; \lambda_0\rangle \,.$$

Further, taking the limit $t \to +\infty$ gives the on-shell states

$$|\boldsymbol{p}, n_-\rangle \equiv \lim_{t \to +\infty} |t, \boldsymbol{p}, n_-\rangle = (2\pi)^{1/2}(2\omega_{\boldsymbol{p}})^{-1}|+\omega_{\boldsymbol{p}}, -\boldsymbol{p}; \lambda_0\rangle \,.$$

We can now reasonably construct Fock spaces $\mathcal{F}_t$ with single time, multiparticle basis states

$$|t; \boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle \equiv |t, \boldsymbol{p}_1, n_{1\pm}; \ldots; t, \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle \,,$$

over all combinations of particle and antiparticle types. Similarly defining

$$|t; \boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}\rangle \equiv |t, \boldsymbol{p}_1, n_{1\pm}; \ldots; t, \boldsymbol{p}_N, n_{N\pm}\rangle \,,$$

we can now take $t \to +\infty$ for particles and antiparticles alike to get the multiparticle on-shell states $|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}\rangle$. The corresponding bra states $\langle \boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}|$ then span a subspace of the dual space $\mathcal{F}'_t$, for any t. Analogously to the case for single particle states, this can be used to define a Hilbert space of multiparticle probability amplitudes for each time t.

Finally, since $|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}\rangle$ now uniformly represents particles and antiparticles in the $t \to +\infty$ limit, it can be used as the asymptotically free state of outgoing particles from a scattering process. The corresponding state for incoming particles is $|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle \equiv \lim_{t \to -\infty} |t; \boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle$.

F. Fields

Even though the theory presented here is essentially first-quantized, it is still often convenient to introduce the formalism of creation and annihilation fields on the Fock space of multi-particle states. Specifically, define the creation field $\hat{\psi}^\dagger(x, n; \lambda)$ by

$$\hat{\psi}^\dagger(x, n; \lambda)|x_1, n_1, \lambda_1; \ldots; x_N, n_N, \lambda_N\rangle = |x, n, \lambda; x_1, n_1, \lambda_1; \ldots; x_N, n_N, \lambda_N\rangle \,,$$

with the corresponding annihilation field $\hat{\psi}(x, n; \lambda)$ having the commutation relation

$$[\hat{\psi}(x', n'; \lambda), \hat{\psi}^\dagger(x, n; \lambda_0)] = \delta_n^{n'} \Delta(x' - x; \lambda - \lambda_0) \, .$$

Further define

$$\hat{\psi}(x, n) \equiv \int_{\lambda_0}^{\infty} \mathrm{d}\lambda \, \hat{\psi}(x, n; \lambda) \, , \tag{45}$$

so that

$$[\hat{\psi}(x', n'), \hat{\psi}^\dagger(x, n; \lambda_0)] = \delta_n^{n'} \Delta(x' - x) \, , \tag{46}$$

which is consistent with the multi-particle inner product as given in Eq. (43).

Note the asymmetry in Eq. (46): $\hat{\psi}^\dagger(x, n; \lambda_0)$ is at the reference value λ_0 of the path parameter (at the start of the path), while in $\hat{\psi}(x', n')$ the path parameter (at the end of the path) is integrated over. This results from the fact that it is the integrated position bra state $\langle x', n'|$, created by $\hat{\psi}(x', n')$, that generates complete particle transition amplitudes (as discussed at the end of Sec. II A). It is thus convenient to consider $\langle x, n|$ to be "dual" to $|x, n; \lambda_0\rangle$, in a similar fashion to the states $\langle \boldsymbol{p}_\pm|$ and $|t, \boldsymbol{p}_\pm; \lambda_0\rangle$ in Sec. II C, even though, by Eq. (21), the position states are not orthogonal.

In the field operator notation, this duality can be captured by introducing a *special adjoint* $\hat{\psi}^\ddagger$ defined by

$$\hat{\psi}^\ddagger(x, n) = \hat{\psi}^\dagger(x, n; \lambda_0) \text{ and } \hat{\psi}^\ddagger(x, n; \lambda_0) = \hat{\psi}^\dagger(x, n) \, . \tag{47}$$

The commutation relation in Eq. (46) then takes on the more symmetric form

$$[\hat{\psi}(x', n'), \hat{\psi}^\ddagger(x, n)] = \delta_n^{n'} \Delta(x' - x) \, .$$

We can also define field operators for explicit particle and antiparticle types, as considered in Sec. II E. Define the *normal particle field* $\hat{\psi}(x, n_+)$ by

$$\hat{\psi}(x, n_+) \equiv \int \mathrm{d}^4 x_0 \, \Delta_+(x - x_0) \hat{\psi}(x_0, n_+; \lambda_0) \, , \tag{48}$$

giving the commutation rule

$$[\hat{\psi}(x', n_+), \hat{\psi}^\dagger(x, n_+; \lambda_0)] = [\hat{\psi}(x', n_+), \hat{\psi}^\ddagger(x, n_+)] = \Delta_+(x' - x) \, . \tag{49}$$

Substituting Eq. (23) into Eq. (48) gives the familiar expression

$$\hat{\psi}(x, n_+) = (2\pi)^{-3/2} \int \mathrm{d}^3 p \, \mathrm{e}^{\mathrm{i}(-\omega_{\boldsymbol{p}} x^0 + \boldsymbol{p} \cdot \boldsymbol{x})} \hat{a}(\boldsymbol{p}, n_+) \, ,$$

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where

$$\hat{a}(\boldsymbol{p}, n_+) \equiv (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1} \int \mathrm{d}^4 x_0 \, e^{\mathrm{i}(\omega_{\boldsymbol{p}} x_0^0 - \boldsymbol{p} \cdot \boldsymbol{x}_0)} \hat{\psi}(x_0, n_+; \lambda_0)$$

is the *on-shell particle 3-momentum field*.

For antiparticles, reverse the roles of the antiparticle creation and annihilation operators relative to increasing-λ propagation as defined for the normal particle type. Define the antiparticle *creation* field analogously to Eq. (48) for the corresponding normal particle *annihilation* field:

$$\hat{\psi}^\dagger(x, n_-) \equiv \int \mathrm{d}^4 x_0 \, \Delta_-(x - x_0) \hat{\psi}^\dagger(x_0, n_-; \lambda_0) \,.$$

Now, $\Delta_-(x - x_0)^* = \Delta_-(x_0 - x)$ (see Eq. (23)). Therefore,

$$\hat{\psi}(x, n_-) = \int \mathrm{d}^4 x_0 \, \Delta_-(x_0 - x) \hat{\psi}(x_0, n_-; \lambda_0) \,, \tag{50}$$

giving the commutation rule (note the switching of x' and x on the right, relative to Eq. (49))

$$[\hat{\psi}(x', n_-), \hat{\psi}^\dagger(x, n_-; \lambda_0)] = [\hat{\psi}(x', n_-), \hat{\psi}^\ddagger(x, n_-)] = \Delta_-(x - x') \,.$$

Substituting Eq. (23) into Eq. (50) and changing variables $\boldsymbol{p} \to -\boldsymbol{p}$ then gives

$$\hat{\psi}(x, n_-) = (2\pi)^{-3/2} \int \mathrm{d}^3 p \, e^{\mathrm{i}(-\omega_{\boldsymbol{p}} x^0 + \boldsymbol{p} \cdot \boldsymbol{x})} \hat{a}(\boldsymbol{p}, n_-) \,,$$

where

$$\hat{a}(\boldsymbol{p}, n_-) \equiv (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1} \int \mathrm{d}^4 x_0 \, e^{\mathrm{i}(\omega_{\boldsymbol{p}} x_0^0 - \boldsymbol{p} \cdot \boldsymbol{x}_0)} \hat{\psi}(x_0, n_-; \lambda_0)$$

is the *on-shell antiparticle 3-momentum field*.

III. INTERACTING PARTICLES

In conventional second-quantized quantum field theory, interactions are introduced via the Lagrangian density into the Hamiltonian used to propagate the fields. The very conception of interacting particles and their paths then only arises at all as a result of the perturbative expansion of the Hamiltonian. Such an approach is thus not very natural for a foundational formalism based on spacetime paths.

Now, the actual traditional motivation for introducing fields in the first place is largely a heuristic response to the well known difficulties with negative energies and probabilities in relativistic quantum mechanics. However, as we have seen in Sec. II, these difficulties

can also be handled in the context of the spacetime path formalism. Further, the spacetime path approach can very directly accommodate the creation and destruction of particles, as required in a relativistic theory. One simply considers particle paths with a finite length: a particle is created at the start of its path and destroyed at the end.

Taking this path viewpoint, an *interaction vertex* can then simply be considered as an event at which a set of particle paths all end together and another set of particle paths all begin. An *interaction graph* is a set of interaction vertices connected by particle paths. For a collection of interacting particles, it is essentially such graphs that act as the fundamental building blocks of the system state, rather than the individual particle paths themselves.

The natural spacetime path approach for interactions is therefore first quantized rather than second quantized. As we will see in this section, the first-quantized spacetime path formalism can duplicate the basic results of perturbative quantum field theory for Feynman diagrams and scattering. It is also consistent with the typically first-quantized geometric approach used in string theory [50].

Of course, taking a first-quantized formalism as foundational requires that issues of consistency and convergence that appear in traditional perturbation theory be addressed directly, without recourse to a posited non-perturbative solution. We will return to this point at the end of Sec. III D, though a full discussion is beyond the scope of the present paper.

A.   Interactions

Since incoming particles are destroyed at an interaction vertex, and outgoing particles are created, the vertex can be represented by an operator constructed as an appropriate product of the creation and annihilation operator fields introduced in Sec. II F. Note that "incoming" and "outgoing" are used here in the sense of the path evolution parameter λ, not time. That is, we are not separately considering particles and antiparticles at this point.

Postulate 6. *An interaction vertex, possibly occurring at any position in spacetime, with some number a of incoming particles and some number b of outgoing particles, is represented by the operator*

$$-i\hat{V} \equiv h \int d^4x \prod_{i=1}^{a} \hat{\psi}^{\ddagger}(x, n_i') \prod_{j=1}^{b} \hat{\psi}(x, n_j), \tag{51}$$

where the coefficient h represents the relative probability amplitude of the interaction and $\hat{\psi}^{\ddagger}$ is the special adjoint defined in Eq. (47).

The probability amplitude for a transition from an initial state $|x_1, n_1; \ldots; x_N, n_N; \lambda_0\rangle$ to a final state $|x'_1, n'_1; \ldots; x'_{N'}, n'_{N'}\rangle$, with a single intermediate interaction, is then

$$
\begin{aligned}
G_1(x'_1, n'_1; \ldots; x'_{N'}, n'_{N'}|x_1, n_1; \ldots; x_N, n_N) \\
= \langle x'_1, n'_1; \ldots; x'_{N'}, n'_{N'}|(-\mathrm{i})\hat{V}|x_1, n_1; \ldots; x_N, n_N; \lambda_0\rangle .
\end{aligned}
$$

This is essentially the amplitude for a first-order *Wick diagram* [51]. That is, it is equivalent to the first-order terms in the Wick expansion of the Dyson series in conventional quantum field theory (including all permutations that may result from crossing symmetries if any of the incoming or outgoing particles in the interaction are of the same type).

The probability amplitude corresponding to multiple intermediate interactions can then be obtained by repeated applications of $\hat{V}$. Thus, the amplitude for m interactions is

$$
\begin{aligned}
G_m(x'_1, n'_1; \ldots; x'_{N'}, n'_{N'}|x_1, n_1; \ldots; x_N, n_N) \\
= \langle x'_1, n'_1; \ldots; x'_{N'}, n'_{N'}|\frac{(-\mathrm{i})^m}{m!}\hat{V}^m|x_1, n_1; \ldots; x_N, n_N; \lambda_0\rangle ,
\end{aligned}
$$

where the $(m!)^{-1}$ factor accounts for all possible permutations of the m identical factors of $\hat{V}$. The *complete* interacting transition amplitude, with any number of intermediate interactions, is then

$$
\begin{aligned}
G(x'_1, n'_1; \ldots; x'_{N'}, n'_{N'}|x_1, n_1; \ldots; x_N, n_N) = \sum_{m=0}^{\infty} G_m(x'_1, n'_1; \ldots; x'_N, n'_N|x_1, n_1; \ldots; x_N, n_N) \\
= \langle x'_1, n'_1; \ldots; x'_N, n'_N|\hat{G}|x_1, n_1; \ldots; x_N, n_N; \lambda_0\rangle ,
\end{aligned}
$$

(52)

where

$$
\hat{G} \equiv \sum_{m=0}^{\infty} \frac{(-\mathrm{i})^m}{m!}\hat{V}^m = \mathrm{e}^{-\mathrm{i}\hat{V}} .
\tag{53}
$$

Extend the operation of the special adjoint in the natural way to sums and products. Then it is clear, at least formally, that $\hat{G}$ is unitary relative to this adjoint (that is, $\hat{G}^{\ddagger}\hat{G} = \hat{G}\hat{G}^{\ddagger} = 1$), so long as $\hat{V}$ is self-adjoint relative to it (that is, $\hat{V}^{\ddagger} = \hat{V}$).

From Eq. (51), there are two consequences to $\hat{V}$ being self-adjoint. First $\mathrm{i}h = g$ must be real. Second, the interaction must involve the same number of incoming and outgoing

particles, of the same types. This second consequence is a result of assuming so far that there is only one possible type of interaction. The formalism can be easily extended to allow for multiple types of interactions by adding additional terms to the definition of $\hat{V}$. In this case, only the overall operator $\hat{V}$ needs to be self-adjoint, not the individual interaction terms.

For example, consider the case of a three-particle interaction of the form $\hat{\psi}^{\ddagger}(x, n_A)\hat{\psi}(x, n_B)\hat{\psi}(x, n_A)$. Then, for the overall interaction operator $\hat{V}$ to be self-adjoint, there must also be a conjugate interaction term $\hat{\psi}^{\ddagger}(x, n_A)\hat{\psi}^{\ddagger}(x, n_B)\hat{\psi}(x, n_A)$. That is,

$$\hat{V} = g \int \mathrm{d}^4 x \left[\hat{\psi}^{\ddagger}(x, n_A)\hat{\psi}(x, n_B)\hat{\psi}(x, n_A) + \hat{\psi}^{\ddagger}(x, n_A)\hat{\psi}^{\ddagger}(x, n_B)\hat{\psi}(x, n_A)\right].$$

This corresponds to the case of the particle of type B being indistinguishable from its antiparticle. Defining the self-adjoint effective field

$$\hat{\psi}'(x, n_B) \equiv \hat{\psi}(x, n_B) + \hat{\psi}^{\ddagger}(x, n_B)$$

then allows $\hat{V}$ to be put back into the form of a single type of interaction:

$$\hat{V} = g \int \mathrm{d}^4 x \, \hat{\psi}^{\ddagger}(x, n_A)\hat{\psi}'(x, n_B)\hat{\psi}(x, n_A) . \tag{54}$$

An alternate interpretation of a self-adjoint interaction vertex is to pair up incoming and outgoing particles of the same type and consider them to be the *same* particle before and after the interaction. For example, an interaction of the form given in Eq. (54) would be considered to represent a *single* particle of type A interacting with a self-adjoint particle of type B.

This viewpoint can be seen more explicitly by considering a first-order interaction matrix element and using Eq. (12) to expand the A-particle propagators:

$$\langle x_A, n_A; x_B, n_B|\hat{V}|x_0, n_A; \lambda_0\rangle$$
$$= g \int \mathrm{d}^4 x \, \Delta_A(x_A - x)\Delta_B(x_B - x)\Delta_A(x - x_0)$$
$$= g \int \mathrm{d}^4 x \, \Delta_B(x_B - x) \int_{\lambda_0}^{\infty} \mathrm{d}\lambda \int_{\lambda_0}^{\infty} \mathrm{d}\lambda' \, \Delta_A(x_A - x; \lambda' - \lambda_0)\Delta_A(x - x_0; \lambda - \lambda_0)$$
$$= g \int \mathrm{d}^4 x \, \Delta_B(x_B - x) \int_{\lambda_0}^{\infty} \mathrm{d}\lambda \int_{\lambda}^{\infty} \mathrm{d}\lambda' \, \Delta_A(x_A - x; \lambda' - \lambda)\Delta_A(x - x_0; \lambda - \lambda_0) .$$

Substituting for the A-particle kernels from Eq. (6), the path integral for the first kernel ends at the same point x as the path integral for the second kernel begins. Therefore, the

two path integrals can be combined into a single path integral, with the constraint that the paths always pass through the intermediate point x:

$$\langle x_A, n_A; x_B, n_B | \hat{V} | x_0, n_A; \lambda_0 \rangle$$

$$= g \int d^4x \, \Delta_B(x_B - x) \int_{\lambda_0}^{\infty} d\lambda \int_{\lambda}^{\infty} d\lambda' \, \eta \int D^4q \, \delta^4(q(\lambda') - x_A)\delta^4(q(\lambda) - x)$$

$$\delta^4(q(\lambda_0) - x_0) e^{iS_A[\dot{q}]}$$

$$= g \int_{\lambda_0}^{\infty} d\lambda' \int_{\lambda_0}^{\lambda'} d\lambda \, \eta \int D^4q \, \delta^4(q(\lambda') - x_A)\delta^4(q(\lambda_0) - x_0) e^{iS_A[\dot{q}]} \Delta_B(x_B - q(\lambda)) \,.$$

This form clearly reflects the viewpoint of a single A-particle interacting with a B-particle at the point $q(\lambda)$ in its path.

Now consider a second-order interaction in which the incoming and outgoing particles are all A-particles:

$$\langle x_1', n_A; x_2', n_A | \frac{1}{2} \hat{V}^2 | x_1, n_A; x_2, n_A; \lambda_0 \rangle$$

$$= \frac{g^2}{2} \int d^4y_1 \int d^4y_2 \, [\Delta_A(x_1' - y_2)\Delta_A(y_2 - y_1)\Delta_A(y_1 - x_1)\Delta_B(y_2 - y_1)\Delta_A(x_2' - x_2)$$

$$+ \Delta_A(x_1' - y_1)\Delta_A(y_1 - x_1)\Delta_B(y_2 - y_1)\Delta_A(x_2' - y_2)\Delta_A(y_2 - x_2) + \cdots] , \quad (55)$$

where the additional terms not shown are the result of position interchanges from the terms given. The first term shown in Eq. (55) reflects a self-interaction of one A particle via the B particle, with the second A particle propagating freely. The self-interaction factor can be given the path integral representation

$$\int d^4y_1 \int d^4y_2 \, \Delta_A(x_1' - y_2)\Delta_A(y_2 - y_1)\Delta_A(y_1 - x_1)\Delta_B(y_2 - y_1)$$

$$= \int_{\lambda_0}^{\infty} d\lambda' \int_{\lambda_0}^{\lambda'} d\lambda_2 \int_{\lambda_0}^{\lambda_2} d\lambda_1 \, \eta \int D^4q \, \delta^4(q(\lambda') - x_1')\delta^4(q(\lambda_0) - x_1) e^{iS_A[\dot{q}]} \Delta_B(q(\lambda_2) - q(\lambda_1)) ,$$

$$(56)$$

reflecting an A particle interacting with the B particle at points λ_1 and λ_2. The second term shown in Eq. (55) reflects an interaction of two A particles via a B particle. It can be

given the path integral representation

$$\int d^4 y_1 \int d^4 y_2 \, \Delta_A(x_1' - y_1) \Delta_A(y_1 - x_1) \Delta_A(x_2' - y_2) \Delta_A(y_2 - x_2) \Delta_B(y_2 - y_1)$$

$$= \int_{\lambda_0}^{\infty} d\lambda_2' \int_{\lambda_0}^{\infty} d\lambda_1' \int_{\lambda_0}^{\lambda_2'} d\lambda_2 \int_{\lambda_0}^{\lambda_1'} d\lambda_1 \, \eta^2 \int D^4 q_2 \, \delta^4(q_2(\lambda_2') - x_2') \delta^4(q_2(\lambda_0) - x_2) e^{iS_A[\dot{q}_2]}$$

$$\times \int D^4 q_1 \, \delta^4(q_1(\lambda_1') - x_1') \delta^4(q_1(\lambda_0) - x_1) e^{iS_A[\dot{q}_1]} \Delta_B(q_2(\lambda_2) - q_1(\lambda_1)),$$

showing the B particle propagating from the point at λ_1 on the path of the first A particle to the point at λ_2 on the path of the second A particle. If the B particle is taken to be a photon, then Barut and Duru have shown that expansions of just the form given above can be obtained from a general path integral formulation of quantum electrodynamics [52] (see also the similar result obtained in [53] using a parametrized perturbation series approach).

B. Feynman Diagrams

Computing a scattering amplitude requires moving from the Wick diagram formulation of Eq. (52) to a Feynman diagram formulation. To do this, replace the initial and final states in Eq. (52) with on-shell multiparticle momentum states $|t_1, \boldsymbol{p}_{1\pm}, n_1; \ldots; t_N, \boldsymbol{p}_{N\pm}, n_N; \lambda_0\rangle$ and $|\boldsymbol{p}_{1\pm}', n_1'; \ldots; \boldsymbol{p}_{N'\pm}', n_{N'}'; \lambda_0\rangle$ (note that these are the on-shell multiparticle states defined in Sec. II E, with antiparticles propagating *backwards* in time, *not* the single-time states defined at the end of that section):

$$G(\boldsymbol{p}_{1\pm}', n_1'; \ldots; \boldsymbol{p}_{N'\pm}', n_{N'}' | \boldsymbol{p}_{1\pm}, n_1; \ldots; \boldsymbol{p}_{N\pm}, n_N)$$

$$\equiv \left[\prod_{i=1}^{N'} 2\omega_{\boldsymbol{p}_i'} \prod_{i=1}^{N} 2\omega_{\boldsymbol{p}_i} \right]^{1/2} \langle \boldsymbol{p}_{1\pm}', n_1'; \ldots; \boldsymbol{p}_{N'\pm}', n_{N'}' | \hat{G} | t_1, \boldsymbol{p}_{1\pm}, n_1; \ldots; t_N, \boldsymbol{p}_{N\pm}, n_N; \lambda_0\rangle. \quad (57)$$

The $2\omega_{\boldsymbol{p}}$ factors are required by the resolution of the identity for these multi-particle states, generalizing the single particle case of Eq. (39):

$$\sum_{N=0}^{\infty} \sum_{n_i\pm} \int d^3 p_1 \cdots d^3 p_N \left[\prod_{i=1}^{N} 2\omega_{\boldsymbol{p}_i} \right]$$

$$\times |t_1, \boldsymbol{p}_{1\pm}, n_1; \ldots; t_N, \boldsymbol{p}_{N\pm}, n_N; \lambda_0\rangle \langle \boldsymbol{p}_{1\pm}, n_1; \ldots; \boldsymbol{p}_{N\pm}, n_N| = 1, \quad (58)$$

where the summation over the $n_i\pm$ is over all particle types *and* particle/antiparticle states.

Note that use of the on-shell states in Eq. (57) requires specifically identifying external lines as particles and antiparticles. For each initial and final particle, $+$ is chosen if it is a normal particle and $-$ if it is an antiparticle. The result is a sum of Feynman diagrams, including all possible permutations of interaction vertices and crossing symmetries. The inner products of the on-shell states for individual initial and final particles with the off-shell states for interaction vertices give the proper factors for the external lines of a Feynman diagram.

For a final particle, the on-shell state $\langle \boldsymbol{p}'_+ |$ is obtained in the limit $t' \to +\infty$. Such a particle is thus an *outgoing* particle from the scattering process. If the external line for this particle starts at an interaction vertex x, then the line contributes an appropriate factor

$$(2\omega_{\boldsymbol{p}'})^{1/2}\langle \boldsymbol{p}'_+ | x; \lambda_0 \rangle = (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}'})^{-1/2}e^{i(+\omega_{\boldsymbol{p}'}x^0 - \boldsymbol{p}'\cdot\boldsymbol{x})} \,.$$

For a final antiparticle, however, the on-shell state $\langle \boldsymbol{p}'_- |$ is obtained in the limit $t' \to -\infty$. This means that the antiparticle is *incoming* to the scattering process, even though it derives from a final vertex, reflecting the time-reversal of antiparticle paths. If the external line for this antiparticle starts at an interaction vertex x, then the line contributes the factor

$$(2\omega_{\boldsymbol{p}'})^{1/2}\langle \boldsymbol{p}'_- | x; \lambda_0 \rangle = (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}'})^{-1/2}e^{i(-\omega_{\boldsymbol{p}'}x^0 + \boldsymbol{p}'\cdot\boldsymbol{x})} \,.$$

Next, consider an initial particle on an external line ending at an interaction vertex x', The factor for this line is (assuming $x'^0 > t$)

$$(2\omega_{\boldsymbol{p}})^{1/2}\langle x' | t, \boldsymbol{p}_+; \lambda_0 \rangle = (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1/2}e^{i(-\omega_{\boldsymbol{p}}x'^0 + \boldsymbol{p}\cdot\boldsymbol{x}')} \,.$$

Note that this expression is independent of t, so we can take $t \to -\infty$ and treat the particle as *incoming*. For an initial antiparticle, the corresponding factor is (assuming $x'^0 < t$)

$$(2\omega_{\boldsymbol{p}})^{1/2}\langle x' | t, \boldsymbol{p}_-; \lambda_0 \rangle = (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1/2}e^{i(+\omega_{\boldsymbol{p}}x'^0 - \boldsymbol{p}\cdot\boldsymbol{x}')} \,.$$

Taking $t \to +\infty$, this represents the factor for an antiparticle that is *outgoing*.

If a particle or antiparticle both starts at an initial vertex x and ends at a final vertex x', then, by Eq. (35),

$$(2\omega_{\boldsymbol{p}'}2\omega_{\boldsymbol{p}})^{1/2}\langle \boldsymbol{p}'_\pm | t, \boldsymbol{p}_\pm; \lambda_0 \rangle = \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \,.$$

Finally, particles that start and end on interaction vertices (i.e., internal edges) are "virtual" particles propagating between interactions, retaining the full Feynman propagator factor $\Delta(x' - x)$.

Thus, the effect of Eq. (57) is to remove the propagator factors from the external lines of the summed Feynman diagrams, retaining them on internal edges. Since, in the position representation, G is essentially a sum of Green's functions G_m, this procedure is effectively equivalent to the usual LSZ reduction of the Green's functions [42, 51, 54].

C. Scattering

The formulation of Eq. (57) is still not that of the usual scattering matrix, since the initial state involves incoming particles but outgoing antiparticles, and vice versa for the final state. To construct the usual scattering matrix, it is necessary to have multiparticle states that involve either all incoming particles and antiparticles (that is, they are composed of individual asymptotic particle states that are all consistently for $t \to -\infty$) or all outgoing particles and antiparticles (with individual asymptotic states all for $t \to +\infty$). These are the states $|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle$ and $|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}\rangle$ defined at the end of Sec. II E.

Reorganizing the scattering amplitude of Eq. (57) in terms of these asymptotic states gives the more usual form using the scattering operator $\hat{S}$. Showing explicitly the asymptotic time limit used for each particle:

$$\langle +\infty, \boldsymbol{p}'_+, n'; \ldots; -\infty, \bar{\boldsymbol{p}}'_-, \bar{n}'; \ldots |\hat{G}| -\infty, \boldsymbol{p}_+, n; \ldots; +\infty, \bar{\boldsymbol{p}}_-, \bar{n}; \ldots; \lambda_0\rangle$$
$$= \langle \boldsymbol{p}', n'_+; \ldots; \bar{\boldsymbol{p}}, \bar{n}_-; \ldots |\hat{S}|\boldsymbol{p}, n_+; \ldots; \bar{\boldsymbol{p}}', \bar{n}'_-; \ldots; \lambda_0\rangle. \quad (59)$$

Using the resolution of the identity

$$\sum_{N=0}^{\infty} \sum_{n_{i\pm}} \int \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_N \left[\prod_{i=1}^{N} 2\omega_{\boldsymbol{p}_i}\right]$$
$$\times |\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle\langle\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}| = 1, \quad (60)$$

expand the state $\hat{S}|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle$ as

$$\hat{S}|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle$$
$$= \sum_{N'=0}^{\infty} \sum_{n_{i\pm}} \int \mathrm{d}^3 p'_1 \cdots \mathrm{d}^3 p'_{N'} \left[\prod_{i=1}^{N'} 2\omega_{\boldsymbol{p}'_i}\right] |\boldsymbol{p}'_1, n'_{1\pm}; \ldots; \boldsymbol{p}'_{N'}, n'_{N'\pm}; \lambda_0\rangle$$
$$\times \langle\boldsymbol{p}'_1, n'_{1\pm}; \ldots; \boldsymbol{p}'_{N'}, n'_{N'\pm}|\hat{S}|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle.$$

This shows how $\hat{S}|\boldsymbol{p}_1, n_{1\pm}; \ldots; \boldsymbol{p}_N, n_{N\pm}; \lambda_0\rangle$ is a superposition of possible out states, with the square of the scattering amplitude, Eq. (59), giving the probability of a particular out state for a particular in state.

Next, use Eqs. (49) and (50) in Eq. (22) to write the propagator as

$$\Delta(x - x_0) = \theta(x^0 - x_0^0)[\hat{\psi}(x, n_+), \hat{\psi}^{\ddagger}(x_0, n_+)] + \theta(x_0^0 - x^0)[\hat{\psi}(x_0, n_-), \hat{\psi}^{\ddagger}(x, n_-)]. \quad (61)$$

Then, reversing the usual derivation for Feynman diagrams (see, for example, [42]) gives the Dyson series expansion

$$\hat{S} = 1 + \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \cdots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n\, \hat{V}(t_1)\hat{V}(t_2)\cdots\hat{V}(t_n) \quad (62)$$

in terms of the *time-dependent interaction operator*

$$\hat{V}(t) \equiv g \int \mathrm{d}^3 x \prod_{i=1}^{a} \hat{\Psi}^{\ddagger}(t, \boldsymbol{x}, n_i') \prod_{j=1}^{b} \hat{\Psi}(t, \boldsymbol{x}, n_j),$$

where

$$\hat{\Psi}(x, n) \equiv \hat{\psi}(x, n_+) + \hat{\psi}^{\ddagger}(x, n_-). \quad (63)$$

Since $\hat{V}(t)$ represents an interaction with the same number of incoming and outgoing particles, of the same types, as $\hat{V}$, the self-adjointness of $\hat{V}$ implies the self-adjointness of $\hat{V}(t)$, from which it can be shown that $\hat{S}$ is unitary. The case of a self-adjoint effective field $\hat{\psi}'(x, n)$ in $\hat{V}$ (as discussed at the end of Sec. III A) corresponds to the requirement of self-adjointness for $\hat{\Psi}(x, n)$. As can be seen from Eq. (63), this requirement implies that particles of type n are indistinguishable from their (path-reversed) antiparticles (indeed, a working definition of "indistinguishable" in this sense might very well be "cannot be distinguished by any interaction").

D. Regularization and Renormalization

Of course, the development given in the previous subsections is actually only formal, because of the usual problems with divergence of the series in Eq. (52). As in conventional field theory, it is necessary to regularize infinite integrals and renormalize the resulting amplitudes. For a first-quantized approach, though, these problems seem particularly severe, since Eq. (52) is taken as the fundamental definition for the interacting amplitude, rather than as a perturbation expansion.

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Fortunately, there is a relatively straightforward way to approach regularization within the context of a spacetime path approach, inspired by the work of Frastai and Horwitz [37] (see also [55, 56] for a similar approach in the context of off-shell electrodynamics). This is to make the interaction coupling dependent on the intrinsic path length. This can be naturally introduced into the spacetime path formalism by making a choice for the weight function $f(T)$ introduced in Sec. II A different than $f(T) = 1$.

To see this, consider that replacing the field operator $\hat{\psi}(x, n)$ defined in Eq. (45) with

$$\hat{\psi}_f(x, n) \equiv \int_{\lambda_0}^{\infty} \mathrm{d}\lambda \, f(\lambda - \lambda_0)\hat{\psi}(x, n; \lambda)$$

gives the commutation relation

$$[\hat{\psi}_f(x', n), \hat{\psi}^\dagger(x, n; \lambda_0)] = \int_0^{\infty} \mathrm{d}T \, f(T)\Delta(x' - x; T) \,,$$

resulting in a propagator including the weight factor $f(T)$. Using this new field operator for, say, particles of type n_A in the interaction vertex operator given in Eq. (54) produces the desired path-length-dependent coupling:

$$\hat{V} = g \int \mathrm{d}^4x \int_{\lambda_0}^{\infty} \mathrm{d}\lambda \, f(\lambda - \lambda_0)\hat{\psi}^\dagger(x, n_A; \lambda_0)\hat{\psi}(x, n_A; \lambda)\hat{\psi}'(x, n_B) \,. \tag{64}$$

For the purposes of the present section, an appropriate choice for $f(\lambda - \lambda_0)$ is the Gaussian

$$f(\lambda - \lambda_0) = \mathrm{e}^{-(\lambda - \lambda_0)^2/2\Delta\lambda^2} \,,$$

where $\Delta\lambda$ is a *correlation length*. For $\Delta\lambda \to \infty$, $f(\lambda - \lambda_0) \to 1$, and Eq. (64) reduces to Eq. (54).

Now, consider again the self-interaction term from Eq. (55). Using the interaction vertex operator from Eq. (64), this becomes

$$\Delta_A(p) \int \mathrm{d}^4p' \int_{\lambda_0}^{\infty} \mathrm{d}\lambda_1 \int_{\lambda_0}^{\infty} \mathrm{d}\lambda_2 \, f(\lambda_2 - \lambda_0)\Delta_A(p'; \lambda_2 - \lambda_0)f(\lambda_1 - \lambda_0)\Delta_B(p - p')\Delta_A(p; \lambda_1 - \lambda_0)$$

For simplicity, the momentum representation has been used here, in which

$$\Delta_A(p; \lambda - \lambda_0) \equiv \mathrm{e}^{-\mathrm{i}(\lambda - \lambda_0)(p^2 + m_A^2)}$$

and

$$\Delta_A(p) \equiv \int_0^{\infty} \mathrm{d}T \, \Delta_A(p; T) = -\mathrm{i}(p^2 + m_A^2 - \mathrm{i}\varepsilon)$$

(and similarly for Δ_B). The propagator from λ_0 to λ_1 is not divergent, so we can let $f(\lambda_1 - \lambda_0) \to 1$, giving $\Delta_A(p)T'(p)\Delta_A(p)$, where

$$T'(p) \equiv \int \mathrm{d}^4 p' \int_{\lambda_0}^{\infty} \mathrm{d}\lambda \, f(\lambda - \lambda_0)\Delta_A(p'; \lambda - \lambda_0)\Delta_B(p - p') \,. \tag{65}$$

Inserting Eq. (13) into into Eq. (65) gives

$$T'(p) = \int \mathrm{d}m^2 \, T(p; m^2)F(m^2) \,, \tag{66}$$

where

$$T(p; m^2) \equiv \int \mathrm{d}^4 p' \, \Delta(p'; m^2)\Delta_B(p - p') \,,$$

is the unregulated self-interaction amplitude (without the external legs), with

$$\Delta(p; m^2) \equiv \int_0^{\infty} \mathrm{d}\lambda' \, \mathrm{e}^{-i\lambda'(p^2 + m^2)} = -i(p^2 + m^2 - i\varepsilon) \,, \tag{67}$$

and

$$F(m^2) \equiv (2\pi)^{-1} \int_0^{\infty} \mathrm{d}\lambda \, \mathrm{e}^{i\lambda(m^2 - m_A^2)} f(\lambda) \,.$$

The unregulated quantity $T(p; m_A^2)$ is divergent. However, Eq. (66) is exactly the Pauli-Villars regularization prescription in continuous form [57]. Adjust the Fourier transform of the coefficients $F(m^2)$ so that

$$\tilde{F}(\lambda) = \begin{cases} f(\lambda)\mathrm{e}^{-i\lambda m_A^2}, & \text{if } \lambda > \delta; \\ 0, & \text{if } \lambda \leq \delta. \end{cases}$$

This then meets the Pauli-Villars conditions in Fourier space for cancelation of singularities [10]: $\tilde{F}(0) = 0$ and $\tilde{F}'(0) = 0$. For $\Delta\lambda \to \infty$ and $\delta \to 0$, $T'(p)$ reduces to the unregulated quantity $T(p; m_A^2)$. (For further discussion, see [37]. In [55, 56], a similar result is obtained for a photon mass spectrum cut-off for the renormalization of off-shell quantum electrodynamics.)

Once the divergent integrals have been regulated, one can apply the usual techniques of multiplicative renormalization in the context of the Feynman diagram formalism obtained in Sec. III B. However, further discussion of renormalization is beyond the scope of the present paper. An intriguing direction for future exploration is the development of a complete regularization and renormalization program based on a physically motivated formulation of spacetime interactions. This would be consistent with the first-quantized approach of

considering the series expansion to be the primary representation of the physical situation of the scattering amplitude, rather than a perturbative approximation to a non-perturbative Lagrangian formulation.

A potentially more serious issue is whether, even after renormalization, series such as that in Eq. (52) converge at all. However, Dyson's classic argument against convergence [58] is based on the conception of traditional quantum electrodynamics, where such series result from perturbation expansion. In the present first-quantized formalism, Dyson's argument might simply imply that the traditional formalism, and arguments from it, are not always applicable.

Actually, it is not the convergence of series for probability amplitudes, such as Eq. (52), that is really important. Rather, the real issue is whether there is a well-defined limit as $N \to \infty$ for physically testable probabilities such as given by

$$\frac{|\langle \alpha_{out} | \hat{S}^{(N)} | \psi_{in} \rangle|^2}{\langle \psi_{in} | \hat{S}^{(N)\ddagger} \hat{S}^{(N)} | \psi_{in} \rangle} ,$$

where $\hat{S}^{(N)}$ is the result of summing Eq. (62) to Nth order, $|\psi_{in}\rangle$ is a properly normalized multiparticle in state and $|\alpha_{out}\rangle$ is a member of a complete basis for multiparticle out states. Quantities such as this for, say, QED produce values that agree with experiment for large N. If it turns out that they do diverge for very large N, this just means that there is some mechanism in the real universe that suppresses the interference effect of interaction graphs with very large N, producing a finite cutoff of the series in Eq. (62).

Indeed, from this perspective, the Lagrangian and Hamiltonian formulations could be viewed as the approximations, obtained by assuming the summing of series for $N \to \infty$. In the end, the problem of divergences might even be seen as an artifact of the conventional second-quantized Lagrangian formulation itself, rather than of its perturbation expansion. Clearly this is an area that bears continued exploration.

IV. CONCLUSION

Spacetime approaches to relativistic quantum mechanics have been developed along a number of different threads in the literature, from the early work on proper time formalisms by Schwinger and others [8, 9, 10], to the equally early work of Stueckelberg [6, 7] and the parametrized relativistic quantum theory it inspired [12, 13, 14, 15, 16, 18, 36], to the

path integral approach introduced by Feynman [4, 11, 17, 52] and its application to quantum gravity [19, 21, 25], to the worldline formalism obtained as the infinite-tension limit of string theory [26, 27, 28, 29, 30, 31, 32] and its relation to the typically first-quantized approach to interaction taken in string theory [50]. The formalism presented in the previous sections can be seen as a foundation underlying all these approaches.

A particularly significant additional result is the derivation of on-shell particles and antiparticle states as the infinite time limit of free particle states. This provides a connection between off-shell parametrized spacetime quantum theories [37, 41, 53, 59, 60] and traditional on-shell quantum field theory. It also suggests the intiguing possibility that, while real particles are likely on-shell to a very high degree of approximation, there may be testable consequences to this approximation not being exact.

The foundation presented here provides a number of interesting avenues for exploration in future publications.

The approach can be readily extended to incorporate path integral representations for non-scalar particles [61, 62, 63, 64, 65]. It can also handle massless particles, though it is not so straightforward to deal properly with the resulting gauge symmetries [59] and non-Abelian interactions.

Further, an important payoff of the spacetime path formalism is the intuitive grounding it gives to the theory, as opposed to the somewhat arbitrary mathematical justifications for introducing fields in traditional quantum field theory. Moreover, the formalism for interacting spacetime paths provides interesting possibilities for addressing the issues of regularization and renormalization (which is all the more important because of the first-quantized nature of the formalism).

Finally, a natural interpretational framework for the formalism is the consistent histories approach to quantum theory [66, 67, 68, 69, 70]. Particle paths can be treated as fine-grained histories in the sense of this approach, with coarse-grained histories corresponding to the superposition of fine-grained states, including cosmological histories of the universe as a whole [19, 71, 72, 73].

For example, scattering probabilities can be considered to represent the probabilities of decohering alternative coarse-grained spacetime histories for the scattering process. Probabilities can even be given to decohering cosmological histories of the universe [74]. Such an interpretation also provides for a natural way to see how the macroscopic classical view

of the universe emerges from the more detailed quantum description, rather than viewing quantum physics as a "quantization" of a classical description (see, for example, [75]), just as one would wish from a foundational quantum theory.

Acknowledgments

APPENDIX A: FORM OF THE PHASE FUNCTIONAL

Proposition. *The phase functional S must have the form*

$$S[\dot{q}] = \int_{\lambda_0}^{\lambda_1} d\lambda' \, L[\dot{q}; \lambda'] \,, \tag{A1}$$

where the parametrization domain for $\dot{q}$ is $[\lambda_0, \lambda_1]$ and $L[\dot{q}; \lambda]$ depends only on $\dot{q}$ and its higher derivatives evaluated at λ.

Proof. In

$$S[\dot{q}; \lambda', \lambda_0] = S[\dot{q}; \lambda', \lambda] + S[\dot{q}; \lambda, \lambda_0] \,,$$

consider $\lambda' = \lambda + \delta\lambda$, for infinitesimal $\delta\lambda$:

$$\begin{aligned}
S[\dot{q}; \lambda + \delta\lambda, \lambda_0] &= S[\dot{q}; \lambda + \delta\lambda, \lambda] + S[\dot{q}; \lambda, \lambda_0] \\
&\approx \delta\lambda \left. \frac{\partial S[\dot{q}; \lambda', \lambda]}{\partial \lambda'} \right|_{\lambda' = \lambda} + S[\dot{q}; \lambda, \lambda_0] \,,
\end{aligned}$$

or

$$\frac{S[\dot{q}; \lambda + \delta\lambda, \lambda_0] - S[\dot{q}; \lambda, \lambda_0]}{\delta\lambda} \approx \left. \frac{\partial S[\dot{q}; \lambda', \lambda]}{\partial \lambda'} \right|_{\lambda' = \lambda} \,.$$

Taking the limit $\delta\lambda \to 0$ then gives

$$\frac{\partial S[\dot{q}; \lambda, \lambda_0]}{\partial \lambda} = L[\dot{q}; \lambda] \,, \tag{A2}$$

where

$$L[\dot{q}; \lambda] \equiv \left. \frac{\partial S[\dot{q}; \lambda', \lambda]}{\partial \lambda'} \right|_{\lambda' = \lambda} \,.$$

Now, the functional L depends only on $\dot{q}$ and λ, not λ_0. Therefore, integrate Eq. (A2) over λ, with the initial condition $S[\dot{q}; \lambda_0, \lambda_0] = 0$, to get

$$S[\dot{q}; \lambda, \lambda_0] = \int_{\lambda_0}^{\lambda} d\lambda' \, L[\dot{q}; \lambda'] \,,$$

which is just Eq. (A1).

Further, by definition $S[\dot{q}; \lambda, \lambda_0]$ only depends on values of $\dot{q}^{\mu}$ between λ_0 and λ. Therefore, $S[\dot{q}; \lambda + \delta\lambda, \lambda] \approx L[\dot{q}; \lambda]\delta\lambda$ should only depend on $\dot{q}$ infinitesimally close to λ. As $\delta\lambda \to 0$, this effectively limits $L[\dot{q}; \lambda]$ to depend only on $\dot{q}$ and its derivatives evaluated at λ.

$\square$

APPENDIX B: EVALUATION OF THE PATH INTEGRAL

Proposition. *The path integral*

$$\Delta(x, x_0; [\lambda]) = \eta[\lambda] \int D^4 q \, \delta^4(q(\lambda(1)) - x)\delta^4(q(\lambda(0)) - x_0) \exp\left(i \int_{\lambda_0}^{\lambda} d\lambda' \left[\frac{1}{4}\dot{q}^2(\lambda') - m^2 \right] \right) . \tag{B1}$$

may be evaluated to get

$$\Delta(x, x_0; [\lambda]) = \Delta(x - x_0; \lambda - \lambda_0) \equiv (2\pi)^{-4} \int d^4 p \, e^{ip \cdot (x - x_0)} e^{-i(\lambda - \lambda_0)(p^2 + m^2)} . \tag{B2}$$

Proof. The path integral in Eq. (B1) may be defined as

$$\Delta(x, x_0; [\lambda]) = \lim_{N \to \infty} \bar{\Delta}^{(N)}(x, x_0; [\lambda]) \,,$$

where

$$\bar{\Delta}^{(N)}(x, x_0; [\lambda]) \equiv \bar{\eta}(\bar{\lambda}_0, \ldots, \bar{\lambda}_N) \int d^4\bar{q}_0 \cdots d^4\bar{q}_N \, \delta^4(\bar{q}_N - x)\delta^4(\bar{q}_0 - x_0)$$

$$\times \exp\left(i \sum_{j=1}^{N} \Delta\bar{\lambda}_j (\frac{1}{4}\dot{\bar{q}}_j^2 - m^2) \right) , \tag{B3}$$

$\bar{\eta}(\bar{\lambda}_0, \ldots, \bar{\lambda}_N) \to \eta[\lambda]$ as $N \to \infty$ and the N-point discrete approximations to the functions $\lambda(s)$ and $q(\lambda(s))$ are given by

$$\bar{\lambda}_j = \lambda(j/N)$$

and

$$\bar{q}_j = q(\bar{\lambda}_j) \,, \tag{B4}$$

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for $j = 0, \ldots, N$. The λ integral is approximated by a summation with

$$\Delta\bar\lambda_j \equiv \bar\lambda_j - \bar\lambda_{j-1}$$

and

$$\dot{\bar q}_j \equiv (\bar q_j - \bar q_{j-1})/\Delta\bar\lambda_j \,, \tag{B5}$$

for $j = 1, \ldots, N$.

To compute the path integral, insert the product of Gaussian integrals

$$\prod_{j=1}^{N} \mathrm{i}\left(\frac{\Delta\bar\lambda_j}{\pi}\right)^2 \int \mathrm{d}^4\bar p_j \, \mathrm{e}^{-\mathrm{i}\Delta\bar\lambda_j \bar p_j^2} = 1$$

into the N-point approximation of Eq. (B3) to get

$$\bar\Delta^{(N)}(x, x_0; [\lambda]) = \bar\xi(\bar\lambda_0, \ldots, \bar\lambda_N) \int \mathrm{d}^4\bar q_0 \cdots \mathrm{d}^4\bar q_N \int \mathrm{d}^4\bar p_1 \cdots \mathrm{d}^4\bar p_N \, \delta^4(\bar q_N - x)\delta^4(\bar q_0 - x_0)$$

$$\times \exp\left(\mathrm{i}\sum_{j=1}^{N}\Delta\bar\lambda_j\left(-\bar p_j^2 + \frac{1}{4}\dot{\bar q}_j^2 - m^2\right)\right),$$

where

$$\bar\xi(\bar\lambda_0, \ldots, \bar\lambda_N) \equiv \left[\prod_{j=1}^{N} \mathrm{i}\left(\frac{\Delta\bar\lambda_j}{\pi}\right)^2\right] \bar\eta(\bar\lambda_0, \ldots, \bar\lambda_N)\,.$$

Inside the $\bar p_j$ integrals, make the change of variables $\bar p_j \to \bar p_j - \frac{1}{2}\dot{\bar q}_j$, so that

$$\sum_{j=1}^{N}\Delta\bar\lambda_j\left(-\bar p_j^2+\frac{1}{4}\dot{\bar q}_j^2-m^2\right) \to \sum_{j=1}^{N}\Delta\bar\lambda_j\left(-\bar p_j^2+\bar p_j\cdot\dot{\bar q}_j-\frac{1}{4}\dot{\bar q}_j^2+\frac{1}{4}\dot{\bar q}_j^2-m^2\right) = \sum_{j=1}^{N}\Delta\bar\lambda_j[\bar p_j\cdot\dot{\bar q}_j-(\bar p_j^2+m^2)]\,. \tag{B6}$$

Now, using Eq. (B5),

$$\sum_{j=1}^{N}\Delta\bar\lambda_j\bar p_j\dot{\bar q}_j = \sum_{j=1}^{N}\bar p_j\cdot(\bar q_j - \bar q_{j-1}) = \bar p_N\cdot\bar q_N - \bar p_1\cdot\bar q_0 - \sum_{j=1}^{N-1}(\bar p_{j+1}-\bar p_j)\cdot\bar q_j$$

(this is essentially just integration by parts within the approximation to the path integral). But, for each $\bar q_j$, $j = 1, \ldots, N-1$,

$$\int \mathrm{d}^4\bar q_j \, \mathrm{e}^{-\mathrm{i}(\bar p_{j+1}-\bar p_j)\cdot\bar q_j} = (2\pi)^4\delta^4(\bar p_{j+1}-\bar p_j)\,,$$

so, integrating over the $\bar p_j$ for $j = 2, 3, \ldots, N$ gives $\bar p_{j+1} = \bar p_j$. Therefore,

$$\bar\Delta^{(N)}(x, x_0; [\lambda]) = \bar\xi(\bar\lambda_0, \ldots, \bar\lambda_N) \int \mathrm{d}^4\bar q_0\, \mathrm{d}^4\bar q_N\, \delta^4(\bar q_N - x)\delta^4(\bar q_0 - x_0)$$

$$\times \int \mathrm{d}^4\bar p_1\,(2\pi)^{4(N-1)}\mathrm{e}^{\mathrm{i}\bar p_1\cdot(\bar q_N - \bar q_0)}\exp\left(-\mathrm{i}\sum_{j=1}^{N}\Delta\bar\lambda_j(\bar p_1^2 + m^2)\right) \tag{B7}$$

$$= (2\pi)^{-4}\bar\zeta(\bar\lambda_0, \ldots, \bar\lambda_N) \int \mathrm{d}^4 p\, \mathrm{e}^{\mathrm{i}p\cdot(x-x_0)}\mathrm{e}^{-\mathrm{i}(\bar\lambda_N - \bar\lambda_0)(p^2+m^2)}\,,$$

where

$$\bar{\zeta}(\bar{\lambda}_0,\ldots,\bar{\lambda}_N) \equiv (2\pi)^{4N}\bar{\xi}(\bar{\lambda}_0,\ldots,\bar{\lambda}_N) = \left[\prod_{j=1}^{N} \mathrm{i}(4\pi\Delta\bar{\lambda}_j)^2\right]\bar{\eta}(\bar{\lambda}_0,\ldots,\bar{\lambda}_N)\,.$$

Now set the normalization factor

$$\bar{\eta}(\bar{\lambda}_0,\ldots,\bar{\lambda}_N) = \prod_{j=1}^{N}(-\mathrm{i})(4\pi\Delta\bar{\lambda}_j)^{-2}\,.$$

Then $\bar{\zeta}(\bar{\lambda}_0,\ldots,\bar{\lambda}_N) = 1$, so we can take the limit $N \to \infty$ of Eq. (B7) to get Eq. (B2). $\quad\square$

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